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DOCTORAL THESIS

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Thresholding Greedy Algorithms in Banach spaces

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Dedicated to my uncle Pablo Manuel, and my grandfather Pablo

"Reserve your right to think, for even to think wrongly is better than not to think at all"

Hypatia of Alexandria

Resumen

Esta tesis trata un tema que combina la Teoría de Aproximación no lineal y el Análisis Funcional: aproximación mediante algoritmos avariciosos (*thresholding greedy algorithms*) en espacios de Banach, un área de investigación popularizada por S. V. Konyagin, V. N. Temlyakov y sus colaboradores, que establecieron las bases de la teoría hace unos 20 años. En el resumen que sigue, revisaremos brevemente la historia de esta área, indicando algunos de los principales resultados, describiremos los problemas que hemos considerado en esta tesis y presentaremos nuestras principales contribuciones, cuyas pruebas y detalles se encontrarán en los capítulos posteriores.

La tesis se divide en seis capítulos que, aparte de algunos preliminares, recopilan una serie de resultados originales que el autor ha obtenido, junto con sus colaboradores, en los últimos 4 años. La mayoría de estos resultados ya han aparecido como artículos publicados en revistas de investigación, y, en algunos casos, los resultados han sido citados por otros autores en la literatura. Algunas contribuciones más recientes forman parte de los preprints que están disponibles en la web y/o se han enviado a revistas. Al final de este resumen, se proporciona una lista de las publicaciones del autor, en las que se basan los resultados de esta tesis.

Introducción histórica

Un problema clásico del Análisis Matemático consiste en la representación de una función f como una suma (infinita)

$$f = \sum_{n=1}^{\infty} a_n \mathbf{e}_n,$$

en términos de una colección dada de “funciones básicas” $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$, y una sucesión de escalares a_n. Algunos ejemplos clásicos son las series de Taylor y de Fourier. En la terminología moderna de Análisis Funcional, se consideran expansiones con respecto a bases (de Schauder), o incluso sistemas más generales, como frames o diccionarios. Por otro lado, uno de los objetivos principales en Teoría de Aproximación es encontrar buenas aproximaciones de un elemento f en términos de sumas finitas de m términos,

$$A_m(f) = \sum_{n\in\Lambda} \lambda_n \mathbf{e}_n,$$

para un conjunto adecuado $\Lambda = \{n_1, \ldots, n_m\}$ y escalares λ_n (posiblemente diferentes de a_n). Se define entonces un *algoritmo de m términos* como un procedimiento determinístico por el cual a cada elemento f y cada $m \geq 1$, se asigna un conjunto Λ de m elementos y coeficientes λ_j como hemos comentado anteriormente. La teoría de aproximación basada en *m términos* busca buenos algoritmos para los cuales el error $\|f - A_m(f)\|$ sea comparable al *mejor error de*

aproximación con m términos:

$$\sigma_m(f) := \inf\Big\{\Big\|f - \sum_{n\in B} b_n \mathbf{e}_n\Big\| \ : \ |B| = m, \ b_n \in \mathbb{F}\Big\}.$$

A lo largo de esta tesis, $\|\cdot\|$ será una norma en un espacio de Banach prefijado $\mathbb{X}$, sobre un cuerpo $\mathbb{F}$. La elección de $\mathbb{X}$ no debe ser arbitraria, puesto que debe contener la clase de elementos f que uno desea estudiar y la norma debe ser la adecuada para esta clase, es decir, valores pequeños de $\|f-g\|$ deben implicar que f y g son "parecidos". Además, supondremos que $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots\}$ será una base de $\mathbb{X}$ de tal forma que a cada $x \in \mathbb{X}$, se le asocia una única expansión

$$x = \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

con $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \dots\}$ los funcionales biortogonales. En la mayoría de los casos, el ejemplo más usual de este tipo de estructura es la base de Schauder, sin embargo, nuestros resultados son válidos en un contexto más general: bases de Markushevich (ver la Definición 1.1.1), donde la anterior expansión es solamente formal, pero la asignación de coeficientes es única. Ejemplos típicos de bases de Markushevich son el sistema trigonométrico y el sistema wavelet en los espacios L_p, $1 \le p < \infty$, y también los ejemplos clásicos de bases en Análisis Funcional.

El segundo ingrediente en esta tesis es el *Thresholding Greedy Algorithm (TGA)*, el cual es un procedimiento de selección de m términos que escoge, para cada $x \in \mathbb{X}$ y $m \ge 1$, el conjunto Λ correspondiente a los m coeficientes de x más grandes en módulo, es decir,

$$G_m(x) = \sum_{n\in\Lambda} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

donde Λ es un conjunto tal que $|\Lambda| = m$ y

$$\min_{n\in\Lambda} |\mathbf{e}_n^*(x)| \ge \max_{n\notin\Lambda} |\mathbf{e}_n^*(x)|.$$

La definición precisa de $(G_m)_{m=1}^{\infty}$ se puede consultar en el Capítulo 2. El TGA fue propuesto por S. V. Konyagin y V. N. Temlyakov en 1999 ([60]), y sus propiedades han sido desarrolladas y estudiadas en otros trabajos posteriores por varios autores como S. J. Dilworth, N. J. Kalton, D. Kutzarova, P. Wojtaszczyk, etc (ver [31, 32, 84]). Algunas contribuciones posteriores, más relacionadas con algunos problemas planteados en esta tesis, han sido llevadas a cabo por F. Albiac, J. L. Ansorena, G. Garrigós, E. Hernández, T. Oikhberg, etc (ver [3, 4, 11, 35, 44]).

Algunos de estos resultados se explican brevemente a continuación y, seguidamente, procederemos a describir las principales contribuciones del autor en esta tesis doctoral, poniendo de manifiesto las diferencias con respecto a trabajos anteriores en la literatura.

Estructura de la tesis y principales resultados

Describimos a continuación los detalles principales de cada uno de los seis capítulos en los que esta tesis está estructurada.

En el **Capítulo 1**, se introducen las herramientas básicas, notaciones y definiciones que se usarán en toda la tesis. Concretamente, estudiaremos la noción de bases de Markushevich, Schauder y bases incondicionales. Adicionalmente, se introducirán las llamadas *funciones de democracia*: con la notación usual $\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n$, con $\varepsilon = (\varepsilon_n)_n$ y $|\varepsilon_n| = 1$,

$$\varphi_u(m) := \sup_{|\varepsilon|=1, |A|\leq m} \|\mathbf{1}_{\varepsilon A}\|, \quad \varphi_l(m) := \inf_{|\varepsilon|=1, |A|\geq m} \|\mathbf{1}_{\varepsilon A}\|.$$

La función φ_u es la *función de democracia por la derecha* y φ_l es la *función de democracia por la izquierda* y ambas funciones juegan un papel fundamental en la teoría de algoritmos greedy. Finalmente, presentamos algunos lemas referentes a convexidad, los cuales estarán presentes en toda la tesis y nos servirán de ayuda para las principales caracterizaciones de bases tipo greedy.

En el **Capítulo 2**, estudiaremos la eficiencia del TGA con respecto a una clase de bases especiales en espacios de Banach. En primer lugar, analizaremos las bases *quasi-greedy*. S. V. Konyagin y V. N. Temlyakov [60] definieron estas bases como aquellas en las que existe una constante positiva C tal que

$$\|x - G_m(x)\| \leq C\|x\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

Más tarde, P. Wojtaszczyk [84], probó el siguiente resultado fundamental en esta teoría:

Teorema ([84])**.** *Una base $\mathscr{B}$ en un espacio de Banach $\mathbb{X}$ es quasi-greedy si y solo si*

$$\lim_{m\to+\infty} \|x - G_m(x)\| = 0, \ \forall x \in \mathbb{X}.$$

Por tanto, asumiendo que una base es quasi-greedy, desde el punto de vista de aproximación, solo estamos suponiendo que el algoritmo converge. Esta es a menudo la hipótesis minimal para el estudio de otras propiedades del algoritmo. En este capítulo, presentaremos una prueba de este teorema que simplifica y corrige algunos pasos de la prueba original y que es válida para bases de Markushevich. Los detalles se encuentran en la Sección 2.1, concretamente en el Teorema 2.1.4, y dicha prueba forma parte de un trabajo original del autor conjuntamente con F. Albiac, J. L. Ansorena y P. Wojtaszczyk en [8].

Seguidamente, presentaremos las llamadas bases *greedy*. S. V. Konyagin y V. N. Temlyakov [60] definieron las bases greedy como aquellas en las que el algoritmo produce la mejor aproximación salvo constante, es decir, existe una constante absoluta C tal que

$$\|x - G_m(x)\| \leq C\sigma_m(x), \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

La mejor constante que verifica esta condición se denota por C_g y se denomina *constante greedy* de $\mathscr{B}$ en $\mathbb{X}$. Un resultado fundamental de S.V. Konyagin y V. N. Temlaykov [60], considerado como el inicio de esta teoría, es el siguiente:

Teorema ([60])**.** *Una base de Markushevich $\mathscr{B}$ en un espacio de Banach $\mathbb{X}$ es greedy si y solo si $\mathscr{B}$ es incondicional y democrática.*

Recordar que $(\mathbf{e}_n)_{n=1}^\infty$ es *democrática* si, denotando por $\mathbf{1}_A = \sum_{n\in A} \mathbf{e}_n$, se tiene que

$$\|\mathbf{1}_A\| \leq C_d\|\mathbf{1}_B\|, \quad \text{para cualesquiera } |A| \leq |B| < \infty,$$

con C_d constante finita y absoluta.

Respecto a este teorema, centramos nuestra atención en estudiar algunas variantes de la prueba original, basadas en ideas similares, produciendo algunas mejoras cuantitativas en la acotación de la constante C_g en términos de las constantes de democracia y de incondicionalidad (ver Teorema 2.3.7). Estas variantes vienen dadas en términos de extensiones de la propiedad de democracia, las cuales se definen en la Sección 2.2. Las técnicas que se usan en este contexto serán de utilidad para capítulos posteriores. Por otro lado, como aplicación especial del Teorema 2.3.7, se recupera el resultado clásico presentado por F. Albiac y P. Wojtaszczyk ([11]), donde los autores caracterizan las bases greedy de constante $C_g = 1$ en términos de la llamada *Propiedad (A)*.

En la Sección 2.3, proporcionamos una nueva caracterización de las bases greedy, la cual se encuentra en un trabajo original del autor y Ó. Blasco ([16]). El Teorema 2.4.2 establece lo siguiente:

Teorema 1. Una base de Markushevich $\mathscr{B}$ es greedy en un espacio de Banach $\mathbb{X}$ si y solo si

$$\|x - G_m(x)\| \leq C\mathscr{D}_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N},$$

donde $\mathscr{D}_m(x) := \inf\{\|x - \alpha \mathbf{1}_{\varepsilon A}\| : |A| = m, |\varepsilon| = 1, \alpha \in \mathbb{F}\}$.

En este caso, $\mathscr{D}_m(x)$ cuantifica el error de aproximación por m términos usando "polinomios con coeficientes de módulo constante". Otras propiedades de este error que forman parte del trabajo [23] son estudiadas en la Subsección 2.4, y dichos resultados son una extensión del [16, Theorem 3.2], donde el autor y Ó. Blasco prueban que para cualquier base ortonormal en un espacio de Hilbert $\mathbb{H}$,

$$\lim_{m\to+\infty} \mathscr{D}_m(x) = \|x\|,\ \forall x \in \mathbb{H}.$$

Concretamente, presentamos un resultado que garantiza cuando $\lim_m \mathscr{D}_m(x)$ se encuentra en un entorno de $\|x\|$ para espacios de Banach generales, usando las funciones de democracia:

Teorema 2. Sea $\mathscr{B}$ una base de Schauder en un espacio de Banach $\mathbb{X}$. Son equivalentes:

i) Existe una constante c tal que

$$c\|x\| \leq \liminf_m \mathscr{D}_m(x) \leq \limsup_m \mathscr{D}_m(x) \leq \|x\|, \text{ para todo } x \in \mathbb{X}.$$

ii) φ_u y φ_l tienen el mismo comportamiento, es decir, o bien ambas divergen, o ambas son acotadas.

Además, si $\varphi_l(m) \to +\infty$ y $\mathscr{B}$ es monótona, entonces

$$\lim_{m\to+\infty} \mathscr{D}_m(x) = \|x\|.$$

En la Sección 2.5, se estudian las llamadas bases *almost-greedy*, esto es, bases en las que existe una constante absoluta C tal que

$$\|x - G_m(x)\| \leq C \inf\left\{\left\|x - \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n\right\| : |A| \leq m\right\},$$

para todo $x \in \mathbb{X}$ y $m \in \mathbb{N}$. Esta noción fue introducida por S. J. Dilworth, N. J. Kalton, D. Kutzarova y V. N. Temlyakov en [32]. Estudiaremos la principal caracterización de estas bases dada en [32], y proporcionaremos alguna simplificación de la prueba proporcionando ciertas mejores cuantitativas sobre las constantes al igual que en las bases greedy (ver el Teorema 2.5.4).

Finalmente, cerraremos este capítulo con la Sección 2.6, donde haremos un repaso sobre las llamadas bases *partially-greedy*, que son aquellas en la que existe una constante C verificando que

$$\|x - G_m(x)\| \leq C \left\| x - \sum_{n=1}^{m} \mathbf{e}_n^*(x)\mathbf{e}_n \right\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

Para estas bases, el TGA siempre se comporta mejor (salvo constante) que el algoritmo lineal basado en las sumas parciales. Por otro lado, discutiremos y proporcionaremos una caracterización para estas bases dada en el Teorema 2.6.2, el cual muestra una pequeña mejora sobre el resultado original dado en [32]. Además, mostraremos el primer ejemplo en la literatura de una base partially-greedy que no es almost-greedy, el cual se encuentra en el trabajo [21].

En el **Capítulo 3**, nos centraremos en la base de Haar, la cual es una de las bases más populares dentro del mundo de las aplicaciones de algoritmos greedy. Particularmente, estudiaremos la dependencia de p para la constante greedy de esta base $\mathscr{H}^{(p)}$ en el espacio $L_p([0,1))$ con $1 < p < \infty$. Dicha constante se denota por $C_g[\mathscr{H}^{(p)}, L_p]$.

Gracias a un resultado de V. N. Temlyakov, el cual fue probado en [77], sabemos que $\mathscr{H}^{(p)}$ es una base greedy en $L_p([0,1))$ para $1 < p < \infty$. Cuando $p = 2$, esta base es ortonormal y la constante greedy es $C_g[\mathscr{H}^{(p)}, L_2] = 1$. Sin embargo, cuando $p \neq 2$, el valor exacto de $C_g[\mathscr{H}^{(p)}, L_p]$ es desconocido. Como la base de Haar es condicional en L_1, la constante $C_g[\mathscr{H}^{(p)}, L_p]$ explota a infinito cuando $p \to 1^+$, y el resultado principal de este capítulo es probar que la constante $C_g[\mathscr{H}^{(p)}, L_p]$ tiene una dependencia lineal con p y explota a infinito en L_1 con orden $p/(p-1)$, contestando así a una pregunta planteada por T. Hytonen ([53]). Dicho resultado (Teorema 3.2.4), es parte de un trabajo conjunto con F. Albiac y J. L. Ansorena ([7]), y es el siguiente.

Teorema 3. Si $1 < p < \infty$ y $p^* = \max\{p, p/(p-1)\}$, entonces

$$C_g[\mathscr{H}^{(p)}, L_p] \approx p^*.$$

Para probar dicho resultado, usaremos de forma novedosa las llamadas bases *bidemocráticas*. Estas bases fueron introducidas en [32], donde los autores querían estudiar si el sistema dual $\mathscr{B}^* = (\mathbf{e}_n^*)_{n=1}^\infty$ en el espacio $\mathbb{X}^*$ conservaba o no la propiedad de ser greedy siempre y cuando $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ fuera greedy en $\mathbb{X}$. Con la notación usual

$$\mathbf{1}_{\varepsilon A} = \sum_{n \in A} \varepsilon_n \mathbf{e}_n, \text{ y } \mathbf{1}_{\varepsilon B}^* = \sum_{n \in B} \varepsilon_n \mathbf{e}_n^*,$$

para $\varepsilon = (\varepsilon_n)$ con $|\varepsilon_n| = 1$, el sistema $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ es *bidemocrático* si existe una constante positiva C_b tal que

$$\|\mathbf{1}_{\varepsilon A}\| \, \|\mathbf{1}_{\varepsilon' B}^*\|_* \leq C_b m, \quad \text{para todo } |A|, |B| \leq m \text{ y } |\varepsilon| = |\varepsilon'| = 1.$$

En la Sección 3.1, analizaremos a fondo esta noción y demostraremos una nueva caracterización de la bidemocracia en el Teorema 3.1.8. Además, en la Proposición 3.1.10, mostraremos que si una base es incondicional y bidemocrática, entonces la base es greedy. Dicho resultado es trivial desde el punto de vista cualitativo, pero gracias a la caracterización anteriormente nombrada, obtendremos una mejora cuantitativa en la acotación de la constante greedy. Concretamente, si K_{su} es la constante de incondicionalidad (ver Definición 1.2), hemos obtenido en [7] el siguiente resultado.

Theorem 4. Si una base de Markushevich en un espacio de Banach $\mathbb{X}$ es K_{su}-incondicional y C_b-bidemocrática, entonces la base es C_g-greedy con

$$C_g \leq K_{su} + C_b.$$

Comparado con los resultados del Capítulo 2, esta acotación superior de C_g es aditiva, lo que nos permite concluir con la acotación lineal en p de la constante $C_g[\mathscr{H}^{(p)}, L_p]$ como hemos resaltado en el Teorema 3.

Finalmente, en la Sección 3.3 de este capítulo, veremos la cualidad de ser greedy de la base de Haar en los espacios ponderados de Lebesgue $L_p(\omega)$. Para ello, consideraremos el espacio discreto de Triebel-Lizorkin $\mathfrak{f}^p(\omega)$, el cual se identifica con $L_p(\omega)$ cuando la base de Haar es incondicional. Estudiaremos entonces que la base canónica en el espacio $\mathfrak{f}^p(\omega)$ es greedy, bajo una condición en el peso ω que introducimos: la *dyadic reverse Carleson condition* (ver Definición 3.3.3), que además nos permite determinar la democracia de dicha base (ver Corolario 3.3.8). Como consecuencia directa, recuperamos que la base de de Haar en $L_p([0,1), \omega)$ es greedy cuando ω pertenece a la clase A_p^d (ver Corolario 3.3.9). Estos resultados forman parte del artículo [17].

En el **Capítulo 4**, nos centraremos en las llamadas *desigualdades tipo Lebesgue* para el TGA. Estas desigualdades, para algoritmos generales de m términos, sirven para cuantificar la eficiencia de dicho algoritmo respecto a la mejor aproximación de m términos.

En el caso específico del Thresholding Greedy Algorithm $(G_m)_m$, desearíamos encontrar, para cada $m = 1, 2, ...$, la constante más pequeña $\mathbf{L}_m$ tal que

$$\|x - G_m(x)\| \leq \mathbf{L}_m \sigma_m(x), \ \forall x \in \mathbb{X}.$$

Observamos que $\mathbf{L}_m$ es acotado si y solo si la base es greedy y, en ese caso, $C_g = \sup_m \mathbf{L}_m$. Para bases que no son greedy, sin embargo, se tiene que $\limsup_m \mathbf{L}_m = \infty$, y queremos entonces estudiar como es el crecimiento de $\mathbf{L}_m$ en términos de algunas propiedades naturales de la base, como incondicionalidad, democracia, etc.

Los primeros resultados que hablan sobre estas desigualdades de tipo Lebesgue para el algoritmo greedy fueron dados bajo la condición de bases quasi-greedy (ver por ejemplo [31], [44], [38], [82] y [35]). Citamos aquí tres resultados que datan de antes de 2013: para un conjunto finito A, usamos la notación

$$P_A(x) = \sum_{n \in A} \mathbf{e}_n^*(x) \mathbf{e}_n.$$

Definimos los parámetros de incondicionalidad y de democracia asociados a la base $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ por

$$k_m = \sup_{|A|\leq m} \|P_A\|, \quad k_m^c = \sup_{|A|\leq m} \|\mathrm{Id}_{\mathbb{X}} - P_A\| \text{ y } \mu_m = \sup_{|A|\leq|B|\leq m} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|}.$$

Teorema ([82]). *Sea $1 < p < \infty$, $p \neq 2$ y $\mathscr{B}$ una base quasi-greedy en L_p. Entonces,*

$$\mathbf{L}_m \leq c_p m^{|1/2-1/p|}, \ \forall m = 1,2,...$$

Teorema ([31, 44]). *Si $\mathscr{B}$ es una base quasi-greedy en un espacio de Banach $\mathbb{X}$, entonces*

$$k_m \lesssim \ln(m), \ \forall m = 1,2,...$$

Teorema ([44]). *Si $\mathscr{B}$ es una base quasi-greedy en un espacio de Banach $\mathbb{X}$, entonces*

$$\mathbf{L}_m \approx \max\{k_m, \mu_m\}, \ \forall m = 1,2,...$$

Nuestra contribución en esta área ha sido proporcionar acotaciones superiores e inferiores de $\mathbf{L}_m$ para bases generales de Markushevich. Los primeros resultados se encuentran en el artículo [18]. Para cada $m \geq 1$, definimos los siguientes parámetros, los cuales hacen referencia a la *super-democracia*, a la *simetría de coeficientes grandes* y a la cualidad de ser quasi-greedy, respectivamente:

$$\tilde{\mu}_m := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, |\varepsilon| = |\varepsilon'| = 1 \right\},$$

$$\nu_m := \sup\left\{ \frac{\|x+\mathbf{1}_{\varepsilon A}\|}{\|x+\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, \sup_j |\mathbf{e}_j^*(x)| \leq 1, A \mathbin{\dot\cup} B \mathbin{\dot\cup} x, |\varepsilon| = |\varepsilon'| = 1 \right\},$$

donde $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ significa que A, B y $\mathrm{supp}(x)$ son disjuntos dos a dos, y

$$g_m := \sup_{G \in \cup_{k\leq m} \mathscr{G}_k} \|G\| \quad \text{and} \quad g_m^c := \sup_{G \in \cup_{k\leq m} \mathscr{G}_k} \|\mathrm{Id}_{\mathbb{X}} - G\|,$$

donde $\mathscr{G}_k$ es la colección de todos los operadores greedy de orden k y $\|G\| = \sup_{x\neq 0} \dfrac{\|G(x)\|}{\|x\|}$. Algunas estimaciones que proporcionamos en [18] son las siguientes (ver Teoremas 4.1.3 y 4.1.5).

Teorema 5. Sea $\mathscr{B}$ una base de Markushevich en un espacio de Banach $\mathbb{X}$. Entonces, para cada $m \geq 1$,

$$\max\{k_m^c \nu_m\} \leq \mathbf{L}_m \leq k_{2m}^c \nu_m.$$

Teorema 6. Sea $\mathscr{B}$ una base de Markushevich en un espacio de Banach $\mathbb{X}$. Entonces, para cada $m \geq 1$,

$$\mathbf{L}_m \leq k_{2m}^c + 2g_m\tilde{\mu}_m.$$

Estos teoremas generalizan los resultados sobre bases greedy (y almost-greedy) del Capítulo 2, que corresponden al caso especial cuando $\sup_m k_m = K_{su}$, $\sup_m \mu_m = C_d$, $\sup_m \tilde{\mu}_m = C_{sd}$, $\sup_m \nu_m = C_a$ y $\sup_m g_m = C_q$ son constantes finitas. Los casos interesantes aparecen cuando algunas de las constantes mencionadas anteriormente son no acotadas, y estos resultados cuantifican como afecta la no acotación de dichos parámetros en el crecimiento de $\mathbf{L}_m$.

En la última parte de la Sección 4.1, introducimos una serie de ejemplos mostrando la optimalidad de los Teoremas 5 y 6, es decir, veremos que para algunas bases de ciertos espacios, las igualdades (o equivalencias asintóticas) de las acotaciones de dichos teoremas se alcanzan (ver Ejemplos 4.1.2, 4.1.11, 4.1.12 y 4.1.14). Entre estos ejemplos, en el Ejemplo 4.1.14, mostraremos la primera base en la literatura que no es quasi-greedy pero que si es incondicional para coeficientes constantes. Todo ello se encuentra en el artículo [18].

En la Sección 4.2, intentaremos dar una nueva perspectiva de este problema. Un inconveniente que tienen los Teoremas 5 y 6 es que las acotaciones superiores son multiplicativas, de tal forma que cuando aplicamos dichas estimaciones a bases que no son quasi-greedy ni democráticas, dichas estimaciones podrían ser no óptimas. Este es el caso, por ejemplo, del sistema trigonométrico en L_p, $1 < p < \infty$, ya que

$$k_m \approx g_m \approx \mu_m \approx \nu_m \approx m^{|\frac{1}{p}-\frac{1}{2}|}.$$

Entonces, el Teorema 5 o 6 no nos permite obtener que $\mathbf{L}_m \approx m^{|\frac{1}{p}-\frac{1}{2}|}$, lo cual fue probado por V. N. Temlyakov en [76]. Describimos a continuación, a groso modo, la nueva técnica desarrollada en la Sección 4.2, la cual se encuentra en el trabajo [19]. Para dos sucesiones positivas w_1 y w_2, definimos la cantidad

$$T_m(w_1,w_2) := \sum_{j=1}^{m} \frac{w_1(j)}{j}\Delta w_2(j),$$

donde $\Delta w(j) = w(j) - w(j-1)$, $j = 1,2,...,$ y

$$\overline{T}_m(w_1,w_2) := \min\{T_m(w_1,w_2), T_m(w_2,w_1)\}.$$

En el Teorema 4.2.16, probamos el siguiente resultado:

Teorema 7. Si $\mathscr{B}$ es una base de Markushevich en un espacio de Banach $\mathbb{X}$, para cada $m \geq 1$,

$$\mathbf{L}_m \leq T_m(\varphi_u, \varphi_u^*),$$

donde φ_u y φ_u^* son las funciones de democracia por la derecha del espacio $\mathbb{X}$ y $\mathbb{X}^*$, respectivamente.

La prueba de este resultado pasa por entender los siguientes embeddings:

$$\ell_{\eta_1}^1 \hookrightarrow \mathbb{X} \hookrightarrow m(\eta_2),$$

donde ℓ_η^1 es un espacio discreto y ponderado de Lorentz y $m(\eta)$ es el espacio discreto de Marcinkiewicz (ver las definiciones en las Secciones 4.2.2 y 4.2.3). Los mejores pesos posibles para estos embeddings resultan ser dados por los Teoremas 4.2.9 y 4.2.14, los cuales dicen lo siguiente:

Teorema 8. Sea $\mathscr{B}$ una base de Markushevich en un espacio de Banach $\mathbb{X}$. Sea w una sucesión positiva. Son equivalentes:

i) $\|\mathbf{1}_{\varepsilon A}\| \leq w(|A|)$ para todo conjunto finito $A \subset \mathbb{N}$ y todo $|\varepsilon| = 1$.

ii) $\ell^1_{\widehat{w}} \hookrightarrow \mathbb{X}$, con $\widehat{w}(j) = j\Delta w(j)$, $j = 1,2,....$

Teorema 9. Sea $\mathscr{B}$ una base de Markushevich en un espacio de Banach $\mathbb{X}$. Sea w una sucesión positiva. Son equivalentes:

(i) $\|\mathbf{1}^*_{\varepsilon A}\|_* \leq w(|A|)$ para todo conjunto finito $A \subset \mathbb{N}$ y todo $|\varepsilon| = 1$.

(ii) $\mathbb{X} \hookrightarrow m(w')$, con $w' = \{j/w(j)\}_{j=1}^{\infty}$.

Estos embeddings nos permiten probar el Teorema 7. En la Sección 4.3, aplicaremos este teorema en varios ejemplos y, además, recuperaremos el comportamiento asintótico de $\mathbf{L}_m$ para el sistema trigonométrico. Entre estos ejemplos destacamos una nueva familia de espacios, denotada por $KT(p,r)$, la cual proporciona una construcción natural de bases quasi-greedy condicionales y que, además, son bidemocráticas (ver Sección 4.3.5). Estas construcciones generalizan el caso $KT(2,2)$ que fue dado por S. V. Konyagin y V. N. Temlyakov en su artículo [60].

En el **Capítulo 5**, nos centraremos en el estudio de otro tipo de algoritmo greedy, el llamado *Thresholding Chebyshev Greedy Algorithm (TCGA)*. Este algoritmo es una variante del TGA, y se define como sigue: dado $x \in \mathbb{X}$ y $m \geq 1$, seleccionamos primero $A = \operatorname{supp}(G_m(x))$. Entonces, encontramos un elemento $\mathfrak{CG}_m(x)$ en el subespacio m-dimensional $[\mathbf{e}_n : n \in A]$ tal que

$$\|x - \mathfrak{CG}_m(x)\| = \operatorname{dist}(x, [\mathbf{e}_n]_{n \in A}) = \min_{a_n \in \mathbb{F}} \Big\| x - \sum_{n \in A} a_n \mathbf{e}_n \Big\|.$$

Para bases 1-incondicionales, siempre se tiene que $\mathfrak{CG}_m(x) = G_m(x)$, pero en general, el algoritmo de Chebyshev produce siempre una mejor selección de coeficientes, lo cual hace que sea mejor para comparar $\|x - \mathfrak{CG}_m(x)\|$ con $\sigma_m(x)$.

El TCGA fue propuesto por S. J. Dilworth, N. J. Kalton y D. Kutzarova en [31], y las principales características de este algoritmo fueron estudiadas allí. En particular, los autores definieron las llamadas bases *semi-greedy*, que son aquellas en las que el TCGA produce la mejor aproximación salvo constante, es decir, existe C tal que

$$\|x - \mathfrak{CG}_m(x)\| \leq C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

La siguiente caracterización también fue probada en [31]:

Teorema ([31])**.** *Si $\mathscr{B}$ es una base de Schauder en un espacio de Banach $\mathbb{X}$ que tiene cotipo finito, entonces $\mathscr{B}$ es semi-greedy si y solo si $\mathscr{B}$ es almost-greedy.*

La primera contribución que damos en este área es quitar la condición de cotipo finito en la anterior caracterización. Además, sustituiremos la condición de Schauder por otra más débil, la cual hemos llamado ρ-admisibilidad (ver la Definición 5.1.1). Esto nos permite incluir varios ejemplos, como bases de Cesàro y sistemas biortogonales con ciertas propiedades (ver el Remark 5.1.3). En la Sección 5.2, obtenemos lo siguiente:

Teorema 10. Sea $\mathbb{X}$ un espacio de Banach y $\mathscr{B}$ una base de Markushevich y ρ-admissible. Entonces $\mathscr{B}$ es semi-greedy si y solo si $\mathscr{B}$ es quasi-greedy y democrática.

Este resultado es el Teorema 5.2.1 donde, además, damos un comportamiento sobre las diferentes constantes que involucran dicho teorema. Estos resultados forman parte del artículo [15] (ver también [22]).

En la última parte de este capítulo, en la Sección 5.3, presentaremos los resultados obtenidos en el trabajo conjunto [20]. Estos resultados recogen el estudio de las desigualdades tipo Lebesgue para el algoritmo TCGA, que estudian la eficiencia de este algoritmo. Definimos entonces, para cada $m = 1,2,...$, la menor constante $\mathbf{L}_m^{\mathrm{ch}}$ tal que

$$\|x - \mathfrak{CG}_m(x)\| \leq \mathbf{L}_m^{\mathrm{ch}} \sigma_m(x),\ \forall x \in \mathbb{X}.$$

Para estudiar $\mathbf{L}_m^{\mathrm{ch}}$, necesitamos los siguientes parámetros que conciernen a la *super-democracia disjunta* y a la *incondicionalidad de coeficientes constantes*:

$$\tilde{\mu}_m^d := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, A \cap B = \emptyset, |\varepsilon| = |\varepsilon'| = 1 \right\},$$

$$\gamma_m := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|} : B \subseteq A, |A| \leq m, |\mathbf{e}_n| = 1 \right\}.$$

En los Teoremas 5.3.1 y 5.3.3, probamos lo siguiente:

Teorema 11. Supongamos que $\mathscr{B}$ es una base de Markushevich en un espacio de Banach $\mathbb{X}$. Para cada $m \geq 1$,

$$\mathbf{L}_m^{\mathrm{ch}} \leq 1 + 2\mathfrak{K} m,$$

donde $\mathfrak{K} = \sup_{n,k} \|\mathbf{e}_n\| \|\mathbf{e}_k^*\|_*$.

Teorema 12. Supongamos que $\mathscr{B}$ es una base de Markushevich en un espacio de Banach $\mathbb{X}$. Para cada $m \geq 1$,

$$\mathbf{L}_m^{\mathrm{ch}} \leq g_{2m}^c + 4\min\{g_m \tilde{\mu}_m, \gamma_{2m} g_{2m} \tilde{\mu}_m^d\}.$$

La optimalidad del Teorema 11 se encuentra en el Ejemplo 5.3.2, donde probamos que la igualdad se alcanza en este resultado. El Teorema 12 proporciona dos diferentes cotas superiores que envuelven los parámetros $\tilde{\mu}_m$ y $\tilde{\mu}_m^d$. La principal razón de ello es porque, en general, para cada $m \in \mathbb{N}$,

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (\tilde{\mu}_m^d)^2,$$

y el cuadrado es esencialmente óptimo como mostramos en el Ejemplo 5.3.7:

Teorema 13. Existe una base de Markushevich en un espacio de Banach $\mathbb{X}$ tal que

$$\limsup_{m \to \infty} \frac{\tilde{\mu}_m}{[\tilde{\mu}_m^d]^{2-\varepsilon}} = \infty,\ \forall \varepsilon > 0.$$

Finalmente, en el **Capítulo 6**, estudiamos una extención del concepto de la aproximación por m términos para el caso ponderado. Este marco de trabajo fue introducido por A. Cohen,

R. A. DeVore y R. Hochmuth en [29] en el contexto de espacios de interpolación, y posteriormente, G. Kerkyacharian, D. Picard y V. N. Temalyakov introdujeron en [58] la noción de bases *w-greedy*. Siendo más preciosos, dada una sucesión positiva $w = (w_n)_{n\geq 1}$, consideramos la siguiente medida en $\mathbb{N}$ dada por

$$w(A) = \sum_{n\in A} w_n, \quad A \subset \mathbb{N}.$$

Para $w_n \equiv 1$ obtenemos la medida de contar, por lo que los nuevos casos corresponden para pesos no constantes. Si $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ es una base de Markushevich en un espacio de Banach $\mathbb{X}$, para cada $t > 0$, consideraremos la siguiente clase de aproximación

$$\Sigma_t^w = \Big\{ \sum_{n\in A} a_n \mathbf{e}_n : w(A) \leq t,\ |A| < \infty,\ a_n \in \mathbb{F} \Big\},$$

y el error de la mejor aproximación ponderada $\sigma_t^w(x) = \operatorname{dist}(x, \Sigma_t^w)$. Entonces, $\mathscr{B}$ es llamada base *w-greedy* cuando

$$\|x - G_m(x)\| \leq C\, \sigma^w_{w(\operatorname{supp}(G_m(x)))}(x), \forall\, x \in \mathbb{X},\ m \geq 1.$$

Uno de los principales resultados en [58] es el siguiente:

Teorema ([58])**.** *Una base $\mathscr{B}$ de Markushevich en un espacio de Banach $\mathbb{X}$ es w-greedy si y solo si $\mathscr{B}$ es incondicional y w-democrática, donde esto último significa que para alguna constante $C > 0$,*

$$\|\mathbf{1}_A\| \leq C\|\mathbf{1}_B\|, \quad \text{para todo par de conjuntos finitos } A, B \text{ con } w(A) \leq w(B).$$

Nuestra contribución es estudiar la extensión de algunos resultados de los capítulos 2 y 5 al caso ponderado, con precisas relaciones que envuelven ciertas constantes (ver Teoremas 6.2.3, 6.2.6, 6.2.8 y 6.2.11). Además, en la Sección 6.3, proporcionamos un ejemplo de una base tipo w-greedy la cual no es greedy en el sentido usual y, además, discutiremos algunas propiedades que este ejemplo preserva dependiendo del comportamiento del peso. Estos resultados se encuentran dentro de los trabajos [21] y [22].

Para finalizar, al final de cada de uno de los capítulos, establecemos una serie de preguntas abiertas que serán interesantes para una investigación futura.

Lista de publicaciones

(1) F. Albiac, J. L. Ansorena, P. M. Berná, *Asymptotic greediness of the Haar system in the spaces $L_p([0,1))$, $1 < p < \infty$.* To appear in Constructive Approximation `https://doi.org/10.1007/s00365-019-09466-1`.

(2) F. Albiac, J. L. Ansorena, P. M. Berná, P. Wojtaszczyk, *Greedy approximation for biorthogonal systems in quasi-Banach spaces*. Submitted (2019). `https://arxiv.org/pdf/1903.11651.pdf`

(3) P. M. Berná, *Equivalence between almost-greedy and semi-greedy bases*. J. Math. Anal. Appl. **470** (2019), no. 1, 218-225.

(4) P. M. Berná, *Characterization of weight-semi-greedy bases*. Submitted (2019). `https://arxiv.org/pdf/1902.10986.pdf`

(5) P. M. Berná, Ó. Blasco, *Characterization of greedy bases in Banach spaces*. J. Approx. Theory, **217** (2017), 28-39.

(6) P. M. Berná, Ó. Blasco, *The best m-th term approximation with respect to polynomials with constant coefficients*. Anal. Math. **43** (2) (2017), 119-132.

(7) P. M. Berná, Ó. Blasco, G. Garrigós, *Lebesgue inequalities for the greedy algorithm in general bases*. Rev. Mat. Complut. **30** (2017), 369-392.

(8) P. M. Berná, Ó. Blasco, G. Garrigós, E. Hernández, T. Oikhberg, *Embeddings and Lebesgue-type inequalities for the greedy algorithm in Banach spaces*, Constr. Approx. **48** (2018), no. 3, 415-451.

(9) P. M. Berná, Ó. Blasco, G. Garrigós, E. Hernández, T. Oikhberg, *Lebesgue-type parameters for the Weak Chebyshev Greedy Algorithm*. Submitted (2019). `https://arxiv.org/abs/1811.04268`

(10) P. M. Berná, S. J. Dilworth, D. Kutzarova, T. Oikhberg, B. Wallis, *The weighted Property (A) and the greedy algorithm*. Submitted (2018). `https://arxiv.org/abs/1803.05052`

(11) P. M. Berná, A. Pérez, *A remark on the approximation with polynomials and greedy bases*. To appear in Journal of Mathematical Analysis and Applications `https://doi.org/10.1016/j.jmaa.2019.05.038`.

Summary

The topics treated in this dissertation lie in the intersection between the theory of Non Linear Approximation and Functional Analysis. More precisely, our main goal during this project has been to provide a better understanding of the so called *Thresholding Greedy Algorithms* in the context of general bases in Banach spaces. This has become a popular research area after S.V. Konyagin, V.N. Temlyakov and their collaborators set the basis of the theory around 20 years ago.

In this summary chapter we shall briefly review the history of this topic, stating some of the main results, and giving a brief overview of the state of the art at the beginning of this project. Then, we shall describe the problems that we have considered in this thesis, and present our main contributions, whose proofs and details are postponed to the subsequent chapters.

This thesis is divided into six chapters, which apart from some preliminaries, compile a number of original results that the author has obtained, together to his collaborators, in the last 4 years. Most of these results have already appeared as published articles in research journals, and in some cases the results have been quoted by other authors in the literature. A few more recent contributions form part of preprints which are available in the web and/or have been submitted to journals. A list of the author's publications, in which the results of this thesis is based, is given at the end of this summary chapter.

Brief description and history

A classical problem in Mathematical Analysis consists in finding representations for a function f as an (infinite) sum

$$f = \sum_{n=1}^{\infty} a_n \mathbf{e}_n,$$

for a given collection of "basic functions" $\{\mathbf{e}_1, \mathbf{e}_2, \dots\}$, and suitable scalars a_n. Classical examples of such representations are the Taylor expansions and the Fourier series of functions. In the modern terminology of Functional Analysis, one considers expansions with regard to a (Schauder) basis, or even to more redundant systems (such as frames, or dictionaries). On the other hand, a main goal in Approximation Theory is to find good approximations to f in terms of *finite sums*, say with m-terms,

$$A_m(f) = \sum_{n \in \Lambda} \lambda_n \mathbf{e}_n,$$

for a suitable set $\Lambda = \{n_1, \dots, n_m\}$ and scalars λ_n (possibly different from a_n). An *m-term algorithm* is a deterministic procedure which to each function f and each $m \geq 1$ assigns a set Λ and coefficients λ_j as above. The theory of *m-term approximation* looks for "good" algorithms (easy to implement, if possible) for which $\|f - A_m(f)\|$ is sufficiently close to the *best m-term* *error of approximation*:

$$\sigma_m(f) := \inf\Big\{ \Big\| f - \sum_{n\in B} b_n \mathbf{e}_n \Big\| \ : \ |B| = m, \ b_n \in \mathbb{F} \Big\}.$$

Here $\|\cdot\|$ denotes the norm in a fixed Banach space $\mathbb{X}$, over a scalar field $\mathbb{F}$. The choice of $\mathbb{X}$ is not arbitrary, as it must contain the class of functions f one wishes to study, while the norm $\|\cdot\|$ should be adequate for this class, in the sense that small values of $\|f-g\|$ must imply "resemblance" between f and g.

In this thesis, the system $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots\}$ is required to be a *basis* of the Banach space $\mathbb{X}$, so that each $x \in \mathbb{X}$ has associated a unique expansion

$$x = \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

with $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \dots\}$ the coefficient functionals. In most usual examples, the system $\mathscr{B}$ is a Schauder basis and the convergence holds in the norm of $\mathbb{X}$. However, our results are also valid in the more general context of *Markushevich bases* (see Definition 1.1.1), in which the expansion above is only formal, but the assignment of coefficients is still unique. This setting includes the most typical examples in applications, such as the trigonometric and wavelet systems in L_p spaces, $1 \le p < \infty$, and also classical examples of bases in Functional Analysis.

The second key ingredient in this thesis is the *Thresholding Greedy Algorithm*, which is a procedure of m-term selection that picks, for each $x \in \mathbb{X}$ and each $m \ge 1$, the set Λ corresponding to the m-largest coefficients $|\mathbf{e}_n^*(x)|$. That is,

$$G_m(x) = \sum_{n\in\Lambda} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

where Λ is a set with $|\Lambda| = m$ such that

$$\min_{n\in\Lambda} |\mathbf{e}_n^*(x)| \ge \max_{n\notin\Lambda} |\mathbf{e}_n^*(x)|,$$

(see the introduction of Chapter 2 for precise definitions). This algorithm was introduced by S. V. Konyagin and V. N. Temlyakov in 1999 ([60]), and their properties developed in a number of subsequent papers by these and other authors, among them S. J. Dilworth, N. J. Kalton, D. Kutzarova, P. Wojtaszczyk, etc (see [31, 32, 84]). Later contributions, more related to the problems that we have considered in this thesis, have been carried out by F. Albiac, J. L. Ansorena, G. Garrigós, E. Hernández, T. Oikhberg, etc (see [3, 4, 11, 35, 44]).

Some of these results are briefly explained in the next subsection. After that, we shall describe the main contributions of the author in this doctoral dissertation, emphasizing the differences with respect to earlier work in the literature.

Structure of the thesis and main results

The present dissertation is structured in six chapters as follows.

In **Chapter 1**, we introduce the basic tools, notations and definitions that we will use throughout this thesis. Concretely, we study the notions of Markushevich, Schauder and unconditional bases. We also introduce the so called *democracy functions*: with the usual notation $\mathbf{1}_{\varepsilon A} = \sum_{n\in A}\varepsilon_n \mathbf{e}_n$, with $\varepsilon = (\varepsilon_n)_n$ and $|\varepsilon_n| = 1$,

$$\varphi_u(m) := \sup_{|\varepsilon|=1,|A|\leq m} \|\mathbf{1}_{\varepsilon A}\|, \quad \varphi_l(m) := \inf_{|\varepsilon|=1,|A|\geq m} \|\mathbf{1}_{\varepsilon A}\|,$$

and φ_u is the *right democracy function* and φ_l is the *left democracy function*. These functions will play a fundamental role in relation with the theory of greedy algorithms. Finally, we state some useful convexity lemmas which will be often present in this dissertation, simplifying many steps in the proofs.

In **Chapter 2**, we study the performance of the Thresholding Greedy Algorithm with respect to special classes of bases in Banach spaces. The first notion that we study is quasi-greediness. S. V. Konyagin and V. N. Temlyakov [60] defined *quasi-greedy bases* as those bases for which there is a positive constant C such that

$$\|x - G_m(x)\| \leq C\|x\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

Later on, P. Wojtaszczyk [84] proved the following fundamental result:

Theorem ([84]). *A basis $\mathscr{B}$ is quasi-greedy if and only if*

$$\lim_{m\to+\infty} \|x - G_m(x)\| = 0, \ \forall x \in \mathbb{X}.$$

Thus, in quasi-greedy bases the greedy algorithm always converges, and therefore this provides the natural (minimal) setting for the study of further properties.

In this dissertation we give a proof of this result, which simplifies and corrects some steps of the original proof, and which is valid for general Markushevich bases. The details are presented in the Section 2.1, concretely in Theorem 2.1.4, and also forms part of a original work of the author with F. Albiac, J. L. Ansorena and P. Wojtaszczyk in [8].

The second notion that we study is greediness. S. V. Konyagin and V. N. Temlyakov [60] defined *greedy bases* as those for which the algorithm produces the best m-approximation up to a constant, that is,

$$\|x - G_m(x)\| \leq C_g \sigma_m(x), \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

The least constant C_g that verifies this condition is called the *greedy constant* of $\mathscr{B}$ in $\mathbb{X}$.

A fundamental result of S.V. Konyagin and V. N. Temlaykov [60], which is usually considered as the starting point in this theory, is the following:

Theorem ([60]). *A Markushevich basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is greedy if and only if $\mathscr{B}$ is unconditional and democratic.*

Recall that $(\mathbf{e}_n)_{n=1}^\infty$ is called *democratic* if, denoting $\mathbf{1}_A = \sum_{n\in A}\mathbf{e}_n$, it holds

$$\|\mathbf{1}_A\| \leq C_d\|\mathbf{1}_B\|, \quad \text{whenever } |A| \leq |B| < \infty,$$

for some finite constant C_d.

Regarding this theorem, we focus our attention in studying variants of the original proof, which based on similar ideas, produce slightly better bounds for the constant C_g in terms of the democracy and unconditionality constants of $\mathscr{B}$ (see Theorem 2.3.7). These involve various notions of democracy-like properties which we define in Section 2.2. The techniques that we use in this stage will be present in many results of other chapters. As a special application of Theorem 2.3.7, we recover the classical result of F. Albiac and P. Wojtaszczyk ([11]) where the authors characterize the bases with greedy constant $C_g = 1$ in terms of the so called *Property (A)*.

In Section 2.3, we give a new characterization of greedy bases, which is an original work of the author with Ó. Blasco ([16]). Namely, in Theorem 2.4.2, we establish the following:

Theorem 1. A Markushevich basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is greedy if and only if

$$\|x - G_m(x)\| \leq C\mathscr{D}_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N},$$

where $\mathscr{D}_m(x) := \inf\{\|x - \alpha \mathbf{1}_{\varepsilon A}\| : |A| = m, |\varepsilon| = 1, \alpha \in \mathbb{F}\}$.

The new functional $\mathscr{D}_m(x)$ quantifies the error of m-term approximation by "polynomials" with coefficients of constant modulus. Further properties of this functional, which are part of the preprint [23], are studied in Subsection 2.4 and the results are an extension of [16, Theorem 3.2], where the author and Ó. Blasco proved that for every orthogonal basis in a Hilbert space $\mathbb{H}$,

$$\lim_{m \to +\infty} \mathscr{D}_m(x) = \|x\|,\ \forall x \in \mathbb{H}.$$

In this case, we show the following result that guarantees when $\lim_m \mathscr{D}_m(x)$ is in a neighborhood of $\|x\|$ using the democracy functions:

Theorem 2. Let $\mathscr{B}$ be a Schauder basis in a Banach space $\mathbb{X}$. The following are equivalent:

i) There exists a positive constant c such that

$$c\|x\| \leq \liminf_m \mathscr{D}_m(x) \leq \limsup_m \mathscr{D}_m(x) \leq \|x\|, \text{ for every } x \in \mathbb{X}.$$

ii) φ_u and φ_l are divergent to infinity or bounded.

Also, if $\varphi_l(m) \to +\infty$ and $\mathscr{B}$ is monotone, then

$$\lim_{m \to +\infty} \mathscr{D}_m(x) = \|x\|.$$

In Section 2.5 we consider *almost-greedy bases*, that is, those bases for which there exists a positive constant C such that

$$\|x - G_m(x)\| \leq C \inf\left\{\left\|x - \sum_{n \in A} \mathbf{e}_n^*(x)\mathbf{e}_n\right\| : |A| \leq m\right\},$$

for all $x \in \mathbb{X}$ and $m \in \mathbb{N}$. This notion was introduced by S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov in [32]. We study the main characterization given in [32], and discuss simplifications in the proof which give slightly better estimates on the constants (see Theorem 2.5.4).

Finally, we close the chapter with Section 2.6, where we briefly review the concept of *partially-greedy bases*, which are those with a positive constant C verifying

$$\|x - G_m(x)\| \leq C\Big\|x - \sum_{n=1}^{m} \mathbf{e}_n^*(x)\mathbf{e}_n\Big\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

For these bases the m-greedy algorithm always performs better (modulo C) than the m-partial sum of the basis. We give a characterization result, Theorem 2.6.2, which slightly improves the original statement in [32]. We also provide the first example in the literature of a partially-greedy basis which is not almost-greedy. This last example is part of the paper [21].

In **Chapter 3**, we study a question regarding the Haar system, which is one the most popular bases in the applications of greedy algorithms. More precisely, we study the dependence on p of the greedy constant $C_g[\mathscr{H}^{(p)}, L_p]$ of the (normalized) Haar system $\mathscr{H}^{(p)}$ in the space $L_p([0,1))$ with $1 < p < \infty$.

In an early result of the theory, V. N. Temlyakov proved in [77] that $\mathscr{H}^{(p)}$ is a greedy basis in $L_p([0,1))$ when $1 < p < \infty$. When $p = 2$, it is an orthonormal basis and the greedy constant $C_g[\mathscr{H}^{(p)}, L_2] = 1$. However, when $p \neq 2$, the exact value of $C_g[\mathscr{H}^{(p)}, L_p]$ is not known. Since the Haar system is not unconditional in L_1 and L_∞, one expects that $C_g[\mathscr{H}^{(p)}, L_p]$ should grow to infinity when $p \to 1^+$ or $p \to \infty$. In the first part of this chapter we prove that the growth rate is linear in p, answering a question that was raised by T. Hytonen ([53]). The concrete result (see Theorem 3.2.4), is part of the paper [7], and can be stated as follows.

Theorem 3. If $1 < p < \infty$ and $p^* = \max\{p, p/(p-1)\}$, then

$$C_g[\mathscr{H}^{(p)}, L_p] \approx p^*.$$

To prove this result we use in a fundamental and novel way the notion of *bidemocratic basis*. This property was introduced in [32] when considering the greediness of the dual system $\mathscr{B}^* = (\mathbf{e}_n^*)_{n=1}^\infty$ in the space $\mathbb{X}^*$. With the usual notation

$$\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n \quad \text{and} \quad \mathbf{1}^*_{\varepsilon B} = \sum_{n\in B} \varepsilon_n \mathbf{e}_n^*,$$

for $\varepsilon = (\varepsilon_n)$ with $|\varepsilon_n| = 1$, a biorthogonal system $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ is called *bidemocratic* if there exists a positive constant C_b such that

$$\|\mathbf{1}_{\varepsilon A}\| \, \|\mathbf{1}^*_{\varepsilon' B}\|_* \leq C_b m, \quad \text{for all } |A|, |B| \leq m \text{ and } |\varepsilon| = |\varepsilon'| = 1.$$

In Section 3.1, we study throughly this notion, and obtain a new characterization in Theorem 3.1.8. Moreover, in Proposition 3.1.10, we show that a basis is greedy if it is unconditional and bidemocratic. Qualitatively, this result is trivial since bidemocracy implies democracy, but we have an improvement with respect to the boundedness constant. Denoting by K_{su} the suppression unconditionality constant (see Definition 1.2), we obtain in [7] the following result:

Theorem 4. If a Markushevich basis in a Banach space is K_{su}-unconditional and C_b-bidemocratic, then the basis is C_g-greedy with

$$C_g \leq K_{su} + C_b.$$

Compared to the results in Chapter 2, this bound of C_g is additive, and allows to derive the linear bound in p asserted in Theorem 3 above.

Finally, in the last Section 3.3 of this chapter, we study the greediness of the Haar system in the weighted Lebesgue spaces $L_p(\omega)$. More precisely, we consider the discrete space of Triebel-Lizorkin type $\mathfrak{f}^p(\omega)$, which identifies with $L_p(\omega)$ when the Haar system is an unconditional basis. We then study the greediness of the canonical basis in this space, under very general conditions in the weight ω. Namely, we introduce a new condition in ω, that we call *dyadic reverse Carleson condition*, (see Definition 3.3.3), that allows us to determine democracy of the basis (see Corollary 3.3.8). As a consequence, we recover that the Haar system in the weighted space $L_p([0,1),\omega)$ is greedy when ω is in the class A_p^d (see Corollary 3.3.9). These results are part of paper [17].

In **Chapter 4**, we turn to the so called *Lebesgue type inequalities* for the greedy algorithm. These inequalities, for a general m-term algorithm, serve to quantify its performance with respect to the best m-term approximation.

In the specific case of the Thresholding Greedy Algorithm $(G_m)_m$, we wish to find, for each $m = 1,2,...$, the smallest value of $\mathbf{L}_m$ such that

$$\|x - G_m(x)\| \leq \mathbf{L}_m \sigma_m(x),\ \forall x \in \mathbb{X}.$$

The parameters $\mathbf{L}_m$ quantify the worst possible performance, over all elements $x \in \mathbb{X}$, of the m-greedy algorithm with respect to the best m-term approximation. Observe that $\mathbf{L}_m$ is bounded if and only if the basis is greedy with $C_g = \sup_m \mathbf{L}_m$. For non-greedy bases, however, we will have $\limsup_m \mathbf{L}_m = \infty$, and we wish to quantify the rate of growth of $\mathbf{L}_m$ in terms of natural properties of the basis, such as the unconditionality, democracy, etc...

The first results about Lebesgue-type inequalities for the TGA were given under the condition that $\mathscr{B}$ is quasi-greedy (see e.g. [31], [44], [38], [82] and [35]). We quote here three such results prior to 2013. We use the notation

$$P_A(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

when A is a finite set. We define the conditionality and democracy constants associated with the basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ by

$$k_m = \sup_{|A|\leq m} \|P_A\|, \quad k_m^c = \sup_{|A|\leq m} \|\mathrm{Id}_{\mathbb{X}} - P_A\| \quad \text{and} \quad \mu_m = \sup_{|A|\leq|B|\leq m} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|}.$$

Theorem ([82])**.** *Let $1 < p < \infty$, $p \neq 2$ and $\mathscr{B}$ a quasi-greedy basis in L_p. Then,*

$$\mathbf{L}_m \leq c_p m^{|1/2-1/p|},\ \forall m = 1,2,...$$

Theorem ([31,44])**.** *If $\mathscr{B}$ is a quasi-greedy basis in a Banach space $\mathbb{X}$, then*

$$k_m \lesssim \ln(m),\ \forall m = 1,2,...$$

Theorem ([44]). *If $\mathscr{B}$ is a quasi-greedy basis in a Banach space $\mathbb{X}$, then*

$$\mathbf{L}_m \approx \max\{k_m, \mu_m\},\ \forall m = 1, 2, ...$$

Our contribution in this topic has been to provide lower and upper bounds of $\mathbf{L}_m$ for general Markushevich bases. The first set of results is contained in the paper [18]. We use the following parameters associated with the super-democracy, the symmetry for largest coefficients and the quasi-greediness:

$$\tilde{\mu}_m := \sup\left\{\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, |\varepsilon| = |\varepsilon'| = 1\right\},$$

$$\nu_m := \sup\left\{\frac{\|x+\mathbf{1}_{\varepsilon A}\|}{\|x+\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, \sup_j |\mathbf{e}_j^*(x)| \leq 1, A \mathbin{\dot\cup} B \mathbin{\dot\cup} x, |\varepsilon| = |\varepsilon'| = 1\right\},$$

where $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ means that A, B and x are pairwise disjoint, and

$$g_m := \sup_{G \in \cup_{k\leq m}\mathscr{G}_k} \|G\| \quad \text{and} \quad g_m^c := \sup_{G \in \cup_{k\leq m}\mathscr{G}_k} \|\mathrm{Id}_{\mathbb{X}} - G\|,$$

where $\mathscr{G}_k$ is the collection of all greedy operators or order k and $\|G\| = \sup_{x\neq 0} \dfrac{\|G(x)\|}{\|x\|}$. Some estimates that we give in [18] are the following (see Theorem 4.1.3 and 4.1.5 below).

Theorem 5. Let $\mathscr{B}$ be a Markushevich basis in a Banach space $\mathbb{X}$. Then, for each $m \geq 1$,

$$\max\{k_m^c \nu_m\} \leq \mathbf{L}_m \leq k_{2m}^c \nu_m.$$

Theorem 6. Let $\mathscr{B}$ be a Markushevich basis in a Banach space $\mathbb{X}$. Then, for each $m \geq 1$,

$$\mathbf{L}_m \leq k_{2m}^c + 2 g_m \tilde{\mu}_m.$$

These theorems generalize the results about greedy (and almost-greedy) bases in Chapter 2, which correspond to the special case when $\sup_m k_m = K_{su}$, $\sup_m \mu_m = C_d$, $\sup_m \tilde{\mu}_m = C_{sd}$, $\sup_m \nu_m = C_a$ and $\sup_m g_m = C_q$ are all finite constants. The new interesting cases appear when some of the sequences $k_m, g_m, \mu_m, \tilde{\mu}_m$ and ν_m are not bounded, and theorems quantify how this affects the growth of $\mathbf{L}_m$.

In the last part of Section 4.1, we give a number of explicit examples that show the optimality of Theorems 5 and 6, in the sense that we can illustrate situations where equalities (or asymptotic equivalences) are attained (see Examples 4.1.2, 4.1.11, 4.1.12 and 4.1.14 below). Among these we shall distinguish Example 4.1.14, as the first instance of a non-quasi-greedy basis which is unconditional for constant coefficients. All the results in Section 4.1 are contained in paper [18].

In Section 4.2, we try to give a different perspective to this problem. One drawback of Theorems 5 and 6 is the multiplicative nature of the estimates, which when applied to bases which are simultaneously not quasi-greedy and not democratic, produce typically non-optimal results. This is the case, for instance, of the trigonometric system in L_p, $1 < p < \infty$, since

$$k_m \approx g_m \approx \mu_m \approx \nu_m \approx m^{|\frac{1}{p}-\frac{1}{2}|}.$$

So, Theorems 5 or 6 will not recover the known result $\mathbf{L}_m \approx m^{|\frac{1}{p}-\frac{1}{2}|}$, proved by V. N. Temlyakov in [76], in another early achievement of this theory.

We now describe the approach taken in Section 4.2, whose results are contained in the paper [19]. For two positive sequences w_1 and w_2, we define the quantity

$$T_m(w_1,w_2) := \sum_{j=1}^{m} \frac{w_1(j)}{j} \Delta w_2(j),$$

where $\Delta w(j) = w(j) - w(j-1)$, $j = 1,2,...$, and

$$\overline{T}_m(w_1,w_2) := \min\{T_m(w_1,w_2), T_m(w_2,w_1)\}.$$

In Theorem 4.2.16, we prove the following result:

Theorem 7. If $\mathscr{B}$ is a Markushevich basis in a Banach space $\mathbb{X}$, for each $m \geq 1$,

$$\mathbf{L}_m \leq T_m(\varphi_u, \varphi_u^*),$$

where φ_u and φ_u^* are the right democracy functions of $\mathbb{X}$ and $\mathbb{X}^*$, respectively.

The proof of this result passes through a careful understanding of the following embeddings,

$$\ell^1_{\eta_1} \hookrightarrow \mathbb{X} \hookrightarrow m(\eta_2),$$

where ℓ^1_η is a discrete weighted Lorentz space and $m(\eta)$ is the discrete Marcinkiewicz space (see precise definitions in Sections 4.2.2 and 4.2.3). The best possible weights for these embeddings turn out to be given by Theorems 4.2.9 and 4.2.14, which can state as follows:

Theorem 8. Let $\mathscr{B}$ be a Markushevich basis in a Banach space $\mathbb{X}$. Let w be a positive sequence. Then, the following are equivalent:

i) $\|\mathbf{1}_{\varepsilon A}\| \leq w(|A|)$ for all finite $A \subset \mathbb{N}$ and all $|\varepsilon| = 1$.
ii) $\ell^1_{\widehat{w}} \hookrightarrow \mathbb{X}$, with $\widehat{w}(j) = j\Delta w(j)$, $j = 1,2,....$

Theorem 9. Let $\mathscr{B}$ be Markushevich basis in a Banach space $\mathbb{X}$, and w a positive sequence. Then, the following are equivalent:

(i) $\|\mathbf{1}^*_{\varepsilon A}\|_* \leq w(|A|)$ for all finite $A \subset \mathbb{N}$ and all $|\varepsilon| = 1$.
(ii) $\mathbb{X} \hookrightarrow m(w')$, with $w' = \{j/w(j)\}_{j=1}^\infty$.

Using these embeddings, we are able to obtain the proof of Theorem 7. In Section 4.3, we test the theorem in numerous examples, and recover in particular the optimal behavior of $\mathbf{L}_m$ for the trigonometric system. Among these examples we outline a new family of spaces, denoted by $KT(p,r)$, which give natural constructions of conditional quasi-greedy bases, that in addition are bidemocratic (see Section 4.3.5). These constructions generalize the case $KT(2,2)$ which was proposed by S. V. Konyagin and V. N. Temlyakov in their celebrated paper [60].

In **Chapter 5**, we turn to the study of the so called *Thresholding Chebyshev Greedy Algorithm*. This algorithm is a variation of the TGA which is defined as follows: given $x \in \mathbb{X}$ and $m \geq 1$, we first select $A = \operatorname{supp}(G_m(x))$. Then we find an element $\mathfrak{CG}_m(x)$ in the m-dimensional subspace $[\mathbf{e}_n : n \in A]$ such that

$$\|x - \mathfrak{CG}_m(x)\| = \operatorname{dist}(x, [\mathbf{e}_n]_{n\in A}) = \min_{a_n \in \mathbb{F}} \Big\| x - \sum_{n\in A} a_n \mathbf{e}_n \Big\|.$$

For 1-unconditional bases one has $\mathfrak{CG}_m(x) = G_m(x)$, but in general, the Chebyshev step produces a better selection of the coefficients, which may be more suited when one compares $\|x - \mathfrak{CG}_m(x)\|$ with $\sigma_m(x)$.

The TCGA was introduced by S. J. Dilworth et al. in [31], and the main properties were studied there. In particular, the authors defined the notion of *semi-greedy basis* as those for which there exists C such that

$$\|x - \mathfrak{CG}_m(x)\| \leq C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

The following characterization was also proved in [31]:

Theorem **([31]).** *If $\mathscr{B}$ is a Schauder basis and $\mathbb{X}$ has finite cotype, then $\mathscr{B}$ is semi-greedy if and only if $\mathscr{B}$ is quasi-greedy and democratic.*

Our first contribution in this area is to remove the finite cotype assumption in the above Theorem. We can also replace the Schauder basis condition by a weaker notion, which we call ρ-admissibility (see Definition 5.1.1 below). This allows to include further examples such as Cesàro bases and biorthogonal systems with certain properties (see Remark 5.1.3). In Section 5.2, we then obtain the following:

Theorem 10. Let $\mathbb{X}$ be a Banach space and $\mathscr{B}$ a Markushevich and ρ-admissible basis. Then $\mathscr{B}$ is semi-greedy if and only if $\mathscr{B}$ is quasi-greedy and democratic.

We refer to Theorem 5.2.1 for a more general result, with precise bounds for the constants. These results form part of the paper [15] (see also [22]).

In the last part of Chapter 5, Section 5.3, we discuss the contents of the paper [20]. Namely, we study the Lebesgue type inequalities for the algorithm $(\mathfrak{CG}_m)_m$ to quantify the performance of the TGCA. We define $\mathbf{L}_m^{\mathrm{ch}}$, for each $m = 1, 2, ...$, as the least number such that

$$\|x - \mathfrak{CG}_m(x)\| \leq \mathbf{L}_m^{\mathrm{ch}} \sigma_m(x),\ \forall x \in \mathbb{X}.$$

To study $\mathbf{L}_m^{\mathrm{ch}}$, we need the following parameters associated with the disjoint-super-democracy and the unconditionality for constant coefficients:

$$\tilde{\mu}_m^d := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} : |A| \leq |B| \leq m, A\cap B = \emptyset, |\varepsilon| = |\varepsilon'| = 1 \right\},$$

$$\gamma_m := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|} : B \subseteq A, |A| \leq m, |\mathbf{e}_n| = 1 \right\}.$$

Then, in Theorems 5.3.1 and 5.3.3, we establish the following:

Theorem 11. Assume that $\mathscr{B}$ is a Markushevich basis in a Banach space $\mathbb{X}$. For each $m \geq 1$,

$$\mathbf{L}_m^{\mathrm{ch}} \leq 1 + 2\mathfrak{K}m,$$

where $\mathfrak{K} = \sup_{n,k} \|\mathbf{e}_n\|\|\mathbf{e}_k\|_*$.

Theorem 12. Assume that $\mathscr{B}$ is a Markushevich basis in a Banach space $\mathbb{X}$. For each $m \geq 1$,

$$\mathbf{L}_m^{\mathrm{ch}} \leq g_{2m}^c + 4\min\{g_m \tilde{\mu}_m, \gamma_{2m} g_{2m} \tilde{\mu}_m^d\}.$$

The optimality of the bound in Theorem 11 can be found in the Example 5.3.2, where we show that the equality holds. Theorem 12 gives two different bounds either involving $\tilde{\mu}_m$ or $\tilde{\mu}_m^d$. The main reason for this is because, in general, for each $m \in \mathbb{N}$,

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (\tilde{\mu}_m^d)^2,$$

and the square bound is essentially optimal as we show in the Example 5.3.7:

Theorem 13. There exists a Markushevich basis in a Banach space $\mathbb{X}$ such that

$$\limsup_{m\to\infty} \frac{\tilde{\mu}_m}{[\tilde{\mu}_m^d]^{2-\varepsilon}} = \infty, \ \forall \varepsilon > 0.$$

Finally, in **Chapter 6**, we study an extension of the concept of m-term approximation to the *weighted* setting. This framework was first introduced by A. Cohen, R. A. DeVore and R. Hochmuth in [29] in the context of interpolation spaces, and later G. Kerkyacharian, D. Picard and V. N. Temalyakov introduced in [58], the notion of w-greedy bases. More precisely, given a positive sequence $w = (w_n)_{n\geq 1}$, we consider the measure in $\mathbb{N}$ given by

$$w(A) = \sum_{n\in A} w_n, \quad A \subset \mathbb{N}.$$

For $w_n \equiv 1$ we obtain the counting measure, so the new cases will correspond to non-constant weights. If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a Markushevich basis of a Banach space $\mathbb{X}$, for each $t > 0$ we consider the approximation class

$$\Sigma_t^w = \Big\{ \sum_{n\in A} a_n \mathbf{e}_n \,:\, w(A) \leq t,\ |A| < \infty,\ a_n \in \mathbb{F} \Big\},$$

and the error of best w-approximation $\sigma_t^w(x) = \mathrm{dist}(x, \Sigma_t^w)$. Then $\mathscr{B}$ is called a *w-greedy basis* when

$$\|x - G_m(x)\| \leq C\,\sigma^w_{w(\mathrm{supp}(G_m(x)))}(x), \forall\, x \in \mathbb{X},\ m \geq 1.$$

One of the main results in [58] is the following:

Theorem **([58]).** *A Markushevich basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is w-greedy if and only if $\mathscr{B}$ is unconditional and w-democratic, the latter meaning that, for some $C > 0$,*

$$\|\mathbf{1}_A\| \leq C\,\|\mathbf{1}_B\|, \quad \text{for all finite } A, B \text{ with } w(A) \leq w(B).$$

Our contribution here is to study the extension of some of the results in Chapters 2 and 5 to the weighted setting, with precise relations about the involved constants (see Theorems

6.2.3, 6.2.6, 6.2.8 and 6.2.11). Also, in Section 6.3, we give an example of a w-greedy-type basis which is not a greedy basis, and we discuss some properties that this example preserves depending on the behavior of the weight. These results are partly contained in paper [21].

We finally remark that at the end of the chapters one can find a number of related open questions, some of which can be interesting topics for future research.

Publication list

(1) F. Albiac, J. L. Ansorena, P. M. Berná, *Asymptotic greediness of the Haar system in the spaces* $L_p([0,1))$, $1 < p < \infty$. To appear in Constructive Approximation `https://doi.org/10.1007/s00365-019-09466-1`.

(2) F. Albiac, J. L. Ansorena, P. M. Berná, P. Wojtaszczyk, *Greedy approximation for biorthogonal systems in quasi-Banach spaces*. Submitted (2019). `https://arxiv.org/pdf/1903.11651.pdf`

(3) P. M. Berná, *Equivalence between almost-greedy and semi-greedy bases*. J. Math. Anal. Appl. **470** (2019), no. 1, 218-225.

(4) P. M. Berná, *Characterization of weight-semi-greedy bases*. Submitted (2019). `https://arxiv.org/pdf/1902.10986.pdf`

(5) P. M. Berná, Ó. Blasco, *Characterization of greedy bases in Banach spaces*. J. Approx. Theory, **217** (2017), 28-39.

(6) P. M. Berná, Ó. Blasco, *The best m-th term approximation with respect to polynomials with constant coefficients*. Anal. Math. **43** (2) (2017), 119-132.

(7) P. M. Berná, Ó. Blasco, G. Garrigós, *Lebesgue inequalities for the greedy algorithm in general bases*. Rev. Mat. Complut. **30** (2017), 369-392.

(8) P. M. Berná, Ó. Blasco, G. Garrigós, E. Hernández, T. Oikhberg, *Embeddings and Lebesgue-type inequalities for the greedy algorithm in Banach spaces*, Constr. Approx. **48** (2018), no. 3, 415-451.

(9) P. M. Berná, Ó. Blasco, G. Garrigós, E. Hernández, T. Oikhberg, *Lebesgue-type parameters for the Weak Chebyshev Greedy Algorithm*. Submitted (2019). `https://arxiv.org/abs/1811.04268`

(10) P. M. Berná, S. J. Dilworth, D. Kutzarova, T. Oikhberg, B. Wallis, *The weighted Property (A) and the greedy algorithm*. Submitted (2018). `https://arxiv.org/abs/1803.05052`

(11) P. M. Berná, A. Pérez, *A remark on the approximation with polynomials and greedy bases*. To appear in Journal of Mathematical Analysis and Applications `https://doi.org/10.1016/j.jmaa.2019.05.038`.

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Contents

Chapter 1

Preliminaries

Throughout this thesis we use standard facts and notation from Banach spaces and approximation theory (see [10]). For the necessary background in the general theory of Banach spaces we refer the reader to [48]. Next we record the notation that is used most heavily.

We write $\mathbb{F}$ for the real or complex scalar field and we assume that $(\mathbb{X}, \|\cdot\|)$ is a Banach space with the norm $\|\cdot\|$ over $\mathbb{F} = \mathbb{C}$ unless otherwise stated. As is customary, we put $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ otherwise. Also, $\langle x_j \colon j \in J \rangle$ stands for the linear span for a family of vector $(x_j)_{j \in J}$ and $[x_j \colon j \in J]$ denotes the closed linear span of $(x_j)_{j \in J}$ in a Banach space $\mathbb{X}$.

A sign will be a scalar of modulus one, and $\operatorname{sign}(\cdot)$ will denote the sign function, i.e., $\operatorname{sign}(0) = 1$ and $\operatorname{sign}(a) = a/|a|$ if $a \in \mathbb{F} \setminus \{0\}$. Also, the conjugate sign is $\overline{\operatorname{sign}(a)} = |a|/a$.

The symbol $\alpha_j \lesssim \beta_j$ for $j \in J$ means that there is a constant $C < \infty$ such that $\alpha_j \leq C\beta_j$ for all $j \in J$. If $\alpha_j \lesssim \beta_j$ and $\beta_j \lesssim \alpha_j$ for $j \in J$ we say $(\alpha_j)_{j \in J}$ are $(\beta_j)_{j \in J}$ are equivalent, and we write $\alpha_j \approx \beta_j$ for $j \in J$.

Other more specific notation will be introduced in context when needed.

1.1 Bases in Banach spaces

Definition 1.1.1. We say that a sequence $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a *Markushevich* basis in a Banach space $\mathbb{X}$ if

(i) $[\mathbf{e}_n \colon n \in \mathbb{N}] = \mathbb{X}$ (completion),

(ii) there is a (unique) sequence $(\mathbf{e}_n^*)_{n=1}^\infty \subset \mathbb{X}^*$, called coordinate functionals (also known as biorthogonal functionals), such that $\mathbf{e}_k^*(\mathbf{e}_n) = \delta_{k,n}$ for all $k, n \in \mathbb{N}$.

(iii) if $\mathbf{e}_j^*(x) = 0$ for all j, then $x = 0$ (totality).

The condition (iii) is equivalent to $[\mathbf{e}_n^* : n \in \mathbb{N}]^{w^*} = \mathbb{X}^*$. Also, we say that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a *semi-normalized Markushevich basis* (SM-basis for short) if $\mathscr{B}$ is Markushevich and

(iv) $\sup_n \|\mathbf{e}_n\| < \infty$ and $\sup_n \|\mathbf{e}_n^*\| < \infty$ (semi-normalization).

We denote by $\mathbf{c}^* := \sup_n \|\mathbf{e}_n^*\|_*$ and $\mathbf{c} := \sup_n \|\mathbf{e}_n\|$.

Remark 1.1.2. Note that (iv) implies that $\inf_n \|\mathbf{e}_n\|$ and $\inf_n \|\mathbf{e}_n^*\|_*$ are not zero. Indeed, we always have that for every $n \in \mathbb{N}$,

$$\|\mathbf{e}_n\| \geq \frac{|\mathbf{e}_n^*(\mathbf{e}_n)|}{\|\mathbf{e}_n^*\|_*}, \ \ \|\mathbf{e}_n^*\|_* \geq \frac{|\mathbf{e}_n^*(\mathbf{e}_n)|}{\|\mathbf{e}_n\|}.$$

Hence, $\|\mathbf{e}_n\| \geq \frac{1}{\mathbf{c}^*}$ and $\|\mathbf{e}_n^*\|_* \geq \frac{1}{\mathbf{c}}$.

If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis, then every $x \in \mathbb{X}$ is univocally determined by its coefficient sequence, i.e., the linear operator

$$\mathscr{F} : \mathbb{X} \to \mathbb{F}^{\mathbb{N}}, \quad x \mapsto (\mathbf{e}_n^*(x))_{n=1}^\infty, \tag{1.1}$$

is one-to-one. The *support* of a vector $x \in \mathbb{X}$ with respect to the basis $\mathscr{B}$ is the set

$$\operatorname{supp}(x) = \{n \in \mathbb{N} \colon \mathbf{e}_n^*(x) \neq 0\},$$

and $|x|_\infty := \sup_{n\in\operatorname{supp}(x)} |\mathbf{e}_n^*(x)|$. Also, we have the following known result.

Lemma 1.1.3. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a Markushevich basis in a Banach space $\mathbb{X}$. Then, $\sup_n \|\mathbf{e}_n^*\|_* < \infty$ if and only if $\mathscr{F}(x) \in c_0$ for all $x \in \mathbb{X}$.

Proof. Given $x \in \mathbb{X}$, let $\varepsilon > 0$ and $z \in \mathbb{X}$ with finite support such that $\|x - z\| < \frac{\varepsilon}{\mathbf{c}^*}$. If $N > N_0 = \max_{i\in\operatorname{supp}(z)} i$,

$$|\mathbf{e}_N^*(x)| = |\mathbf{e}_N^*(x-z)| \leq \mathbf{c}^*\|x-z\| < \varepsilon.$$

Hence, $\mathscr{F}(x) \in c_0$. For the converse, let $K_x := \sup_{n\in\mathbb{N}} |\mathbf{e}_n^*(x)| < \infty, \forall x \in \mathbb{X}$. Then, by the Uniform Boundedness Principle, $\sup_n \|\mathbf{e}_n^*\|_* < \infty$. □

1.1.1 Linear operators associated to bases

Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$. For every finite set $A \subset \mathbb{N}$, we define the *projection operator* P_A as:

$$P_A : \mathbb{X} \to \langle \mathbf{e}_i : i \in A \rangle, \quad x \mapsto \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n.$$

We also use the notation $P_{A^c} := \mathrm{Id}_{\mathbb{X}} - P_A$. $\mathscr{B}$ is said to be *(suppression) unconditional* with constant K_{su} or K_{su}-unconditional, if

$$K_{su}[\mathscr{B},\mathbb{X}] := K_{su} = \sup_{\substack{A\subset\mathbb{N} \\ finite}} \|P_A\| < \infty. \tag{1.2}$$

Remark 1.1.4. If $\mathscr{B}$ is an SM-unconditional basis with constant K_{su}, then, for every finite set $A \subset \mathbb{N}$,

$$\|x - P_A(x)\| \leq K_{su}\|x\|. \tag{1.3}$$

Indeed, if $x \in \mathbb{X}$, for each $\varepsilon > 0$, there exists an element with finite support $y \in \mathbb{X}$ such that $\|x-y\| < \varepsilon$. Hence,

$$\begin{aligned}\|x - P_A(x)\| = \|P_{A^c}(x)\| &= \|P_{A^c}(x) - P_{A^c \cap \operatorname{supp}(y)}(y) + P_{A^c \cap \operatorname{supp}(y)}(y)\| \\ &\leq \|P_{A^c}\|\|x-y\| + K_{su}\|y\| \\ &\leq (\|P_{A^c}\| + K_{su})\varepsilon + K_{su}\|x\|.\end{aligned}$$

Letting ε tend to 0, we obtain (1.3).

For Banach spaces, there are several characterizations of unconditional bases and here we present one of them. We say that an SM-basis $\mathscr{B}$ is *lattice unconditional* if there exists a constant K such that

$$\left\|\sum_{n=1}^N a_n \mathbf{e}_n\right\| \leq K \left\|\sum_{n=1}^N b_n \mathbf{e}_n\right\|, \quad \text{for all } |a_n| \leq |b_n| \text{ and all } N \in \mathbb{N}. \tag{1.4}$$

We denote by $K_u[\mathscr{B},\mathbb{X}] := K_u$ the least constant that verifies (1.4) and we will say that $\mathscr{B}$ is K_u-lattice-unconditional. Also, it is easy to prove (see for instance [10, page 54] for the real case) that

$$K_{su} \leq K_u \leq 2\kappa K_{su}, \tag{1.5}$$

where $\kappa = 1$ if $\mathbb{F} = \mathbb{R}$ and $\kappa = 2$ if $\mathbb{F} = \mathbb{C}$.

A nice property of unconditional bases that will appear again in Chapter 2 is the following.

Proposition 1.1.5. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ an SM-basis in a Banach space $\mathbb{X}$. If $\mathscr{B}$ is K_u-lattice unconditional, then, for any finite collection $(a_n)_{n\in A}$ with $A \subset \mathbb{N}$ a finite set,

$$\frac{1}{K_u}\min_{n\in A}|a_n| \left\|\sum_{n\in A} \mathbf{e}_n\right\| \leq \left\|\sum_{n\in A} a_n \mathbf{e}_n\right\| \leq K_u \max_{n\in A}|a_n| \left\|\sum_{n\in A} \mathbf{e}_n\right\|. \tag{1.6}$$

Proof. The proof is a direct consequence of (1.4). □

A system $(\mathbf{e}_n)_{n=1}^\infty$ in a Banach space $\mathbb{X}$ is said to be a *Schauder basis* if for every $x \in \mathbb{X}$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that the series $\sum_{n=1}^\infty a_n \mathbf{e}_n$ converges to x. It is known that every Schauder basis with $0 < \inf_n \|\mathbf{e}_n\| \leq \sup_n \|\mathbf{e}_n\| < \infty$ is an SM-basis (see e.g. [10, Theorem 1.1.3]). The partial-sum projections associated to $(\mathbf{e}_n)_{n=1}^\infty$ are denoted by

$$P_m := P_{\{1,\dots,m\}}, \quad m \in \mathbb{N}.$$

It is well known (see [10, Proposition 1.1.4]) that an SM-basis $\mathscr{B}$ is Schauder if and only if

$$\mathfrak{K}_b[\mathscr{B},\mathbb{X}] := \mathfrak{K}_b = \sup_m \|P_m\| < \infty. \tag{1.7}$$

We will refer to $\mathfrak{K}_b[\mathscr{B},\mathbb{X}]$ as the *basis constant* of $\mathscr{B}$ in $\mathbb{X}$.

1.2 Democracy functions

In this section we study some properties about the *democracy functions*. To define them, we need to consider the *indicator sums*. Given a finite set $A \subset \mathbb{N}$, we shall denote

$$\Psi_A := \{\varepsilon = (\varepsilon_j)_{j\in A} : |\varepsilon_j| = 1\}.$$

Given an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$, for a finite set A and $\varepsilon \in \Psi_A$, we define the indicator sums as

$$\mathbf{1}_{\varepsilon A}[\mathscr{B},\mathbb{X}] := \mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n.$$

For $\varepsilon \equiv 1$, we use the notation $\mathbf{1}_A$.

Definition 1.2.1. The *upper democracy function* is defined as

$$\varphi_u[\mathscr{B},\mathbb{X}](m) := \varphi_u(m) = \sup_{\varepsilon\in\Psi_A, |A|\leq m} \|\mathbf{1}_{\varepsilon A}\|,$$

and the *lower democracy function* is defined as

$$\varphi_l[\mathscr{B},\mathbb{X}](m) := \varphi_l(m) = \inf_{\varepsilon\in\Psi_A, |A|\geq m} \|\mathbf{1}_{\varepsilon A}\|.$$

Below, we shall also consider the function

$$h_l[\mathscr{B},\mathbb{X}](m) := h_l(m) = \inf_{\varepsilon\in\Psi_A, |A|= m} \|\mathbf{1}_{\varepsilon A}\|.$$

Some known properties of these functions are the following.

Lemma 1.2.2. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) $\frac{1}{\mathbf{c}^*} \leq \varphi_l(m) \leq \varphi_u(m) \leq \mathbf{c}m$.

b) φ_u and φ_l are non-decreasing and $(\varphi_u(m)/m)_m$ is non-increasing.

c) φ_u is doubling, that is, there exists a constant $d > 0$ such that $\varphi_u(2m) \leq d\varphi_u(m)$ for every $m \in \mathbb{N}$.

d) $\varphi_u(m) = \sup_{|A|=m, \varepsilon\in\Psi_A} \|\mathbf{1}_{\varepsilon A}\|$.

Proof. The properties are easy to verify, and can be found, for instance, in [10], [32] and [65]. □

Remark 1.2.3. If $\mathscr{B}$ is an SM-Schauder basis with basis constant $\mathfrak{K}_b$,

$$\varphi_l(m) \leq h_l(m) \leq \mathfrak{K}_b \varphi_l(m).$$

Moreover, the right hand inequality may not be true for non-Schauder bases. T. Oikhberg (personal communication) found a non-Schauder basis for which $\varphi_l(m)$ and $h_l(m)$ are not comparable, that is,

$$\sup_m \frac{h_l(m)}{\varphi_l(m)} = \infty.$$

Finally, we also remark that φ_l may in general not be doubling as P. Wojtaszczyk proved in [87].

1.3 Convexity

One of the main tools that we will use in this thesis is the convexity. We present the following lemma that can be found, for instance, in [18, Lemma 2.7]. As usual, the convex hull of a set S is defined as

$$\operatorname{co}(S)=\left\{\sum_{j=1}^{n}\lambda_j s_j : s_j\in S, 0\le\lambda_j\le 1, \sum_{j=1}^{n}\lambda_j=1, n\in\mathbb{N}\right\}.$$

Lemma 1.3.1. Assume that $\mathbb{X}$ is a Banach space, $J\subset\mathbb{N}$ finite and $\{f_j\}_{j\in J}\subset\mathbb{X}$. Then

$$\operatorname{co}\left\{\sum_{n\in J}\varepsilon_n f_n : |\varepsilon_n|=1\right\}=\left\{\sum_{n\in J}z_n f_n : |z_n|\le 1\right\}.$$

Proof. We sketch the proof in the complex case, where it may be less obvious. The inclusion "$\subseteq$" is clear, since each $\sum_{n\in J}\varepsilon_n f_n$ belongs to the set R on the right hand side, and R is a convex set. To show "$\supseteq$" one proceeds by induction in $N=|J|$. It is clear for $N=1$, so we show the case N from the case $N-1$. We may assume that $J=\{1,\dots,N\}$. Pick any $z=\sum_{n=1}^{N}z_n f_n\in R$, that is $|z_n|\le 1$. Write $z_N=re^{i\theta}$, and by the induction hypothesis

$$z'=\sum_{n=1}^{N-1}z_n f_n=\sum_{\varepsilon}\lambda_\varepsilon\left(\varepsilon_1 f_1+\ldots+\varepsilon_{N-1}f_{N-1}\right),$$

for suitable numbers $0\le\lambda_e\le 1$ such that $\sum_\varepsilon\lambda_\varepsilon=1$. Then we have

$$\begin{aligned} z &= \tfrac{1+r}{2}\left[z'+e^{i\theta}f_N\right]+\tfrac{1-r}{2}\left[z'-e^{i\theta}f_N\right]\\ &= \sum_{\varepsilon,\pm}\tfrac{1\pm r}{2}\lambda_\varepsilon\left(\varepsilon_1 f_1+\ldots+\varepsilon_{N-1}f_{N-1}\pm e^{i\theta}\varepsilon_N\right), \end{aligned}$$

which belongs to the set on the left hand side. □

Corollary 1.3.2. Let $\mathbb{X}$ be a Banach space. If $g\in\mathbb{X}$, $J\subset\mathbb{N}$ a finite set, $\{f_j\}_{j\in J}\subset\mathbb{X}$ and $|a_j|\le 1$ for $j\in J$, then

$$\left\|g+\sum_{j\in J}a_j f_j\right\|\le\sup_{\varepsilon\in\Psi_J}\left\|g+\sum_{j\in J}\varepsilon_j f_j\right\|. \tag{1.8}$$

Proof. Let $n_0\notin J$ and define $J'=J\cup\{n_0\}$. Define now $\tilde{f}_j$ and $\tilde{a}_j$, $j\in J'$, as follows:

- $\tilde{f}_j=f_j$ and $\tilde{a}_j=a_j$ if $j\in J$.
- $\tilde{f}_j=g$ and $\tilde{a}_j=1$ if $j=n_0$.

Then, applying Lemma 1.3.1, there exists a collection λ_ε with $0 \leq \lambda_\varepsilon \leq 1$ and $\sum_\varepsilon \lambda_\varepsilon = 1$, such that

$$\begin{aligned}
\left\| g + \sum_{j\in J} a_j f_j \right\| &= \left\| \sum_{j\in J'} \tilde{a}_j \tilde{f}_j \right\| = \left\| \sum_{\varepsilon\in\Psi_{J'}} \lambda_\varepsilon \left(\sum_{j\in J'} \varepsilon_j \tilde{f}_j \right) \right\| \\
&\leq \sum_{\varepsilon\in\Psi_{J'}} \lambda_\varepsilon \left\| \sum_{j\in J'} \varepsilon_j \tilde{f}_j \right\| = \sum_{\varepsilon\in\Psi_{J'}} \lambda_\varepsilon \left\| \varepsilon_{n_0} g + \sum_{j\in J} \varepsilon_j f_j \right\| \\
&= \sum_{\varepsilon\in\Psi_{J'}} \lambda_\varepsilon |\varepsilon_{n_0}| \left\| g + \sum_{j\in J} \overline{\varepsilon}_{n_0} \varepsilon_j f_j \right\| \\
&\leq \sup_{\varepsilon\in\Psi_J} \left\| g + \sum_{j\in J} \varepsilon_j f_j \right\|.
\end{aligned}$$

□

Corollary 1.3.3. Let $\mathbb{X}$ be a Banach space. If $g \in \mathbb{X}$, J is a finite set $J \subset \mathbb{N}$ and $\{f_j\}_{j\in J} \subset \mathbb{X}$, then:

(i) $\|\sum_{j\in J} a_j f_j\| \leq \max_{j\in J} |a_j| \sup_{\varepsilon\in\Psi_J} \|\sum_{j\in J} \varepsilon_j f_j\|$.

(ii) $\|\sum_{j\in J} a_j f_j\| \leq 2\kappa(\max_{j\in J} |a_j|) \sup_{D\subseteq J} \|\sum_{j\in D} f_j\|$, where $\kappa = 1$ if $\mathbb{F} = \mathbb{R}$ and $\kappa = 2$ if $\mathbb{F} = \mathbb{C}$.

Proof. To show (i), we apply Corollary 1.3.2 to the sequence $(b_j)_{j\in J}$, where $b_j = a_j/t$, with $t := \max_{j\in J} |a_j|$ and with $g = 0$.

To show (ii), we proceed as follows. Let $\mathbb{F} = \mathbb{R}$, $\varepsilon_j \in \{\pm 1\}$ and define the sets $J^\pm := \{j \in J : \varepsilon_j = \pm 1\}$. Then,

$$\left\| \sum_{j\in J} \varepsilon_j f_j \right\| \leq \left\| \sum_{j\in J^+} f_j \right\| + \left\| \sum_{j\in J^-} f_j \right\| \leq 2 \sup_{D\subseteq J} \left\| \sum_{j\in D} f_j \right\|. \tag{1.9}$$

Hence, applying now the case (i), we obtain the result for the real case. Now, for the complex case,

$$\left\| \sum_{j\in J} \varepsilon_j f_j \right\| \leq \left\| \sum_{j\in J} \mathrm{Re}(\varepsilon_j) f_j \right\| + \left\| \sum_{j\in J} \mathrm{Im}(\varepsilon_j) f_j \right\|.$$

Applying (i) in the last inequality,

$$\left\| \sum_{j\in J} \mathrm{Re}(\varepsilon_j) f_j \right\| + \left\| \sum_{j\in J} \mathrm{Im}(\varepsilon_j) f_j \right\| \leq 2 \sup_{\varepsilon_j=\pm 1} \| \sum_{j\in J} \varepsilon_j f_j \|. \tag{1.10}$$

Now, if we use (1.9) in the right hand inequality of (1.10),

$$\left\| \sum_{j\in J} \varepsilon_j f_j \right\| \leq 4 \sup_{D\subseteq J} \left\| \sum_{j\in D} f_j \right\|. \tag{1.11}$$

Finally, applying (i),

$$\left\|\sum_{j\in J} a_j f_j\right\| \le 4(\max_{j\in J}|a_j|)\sup_{D\subseteq J}\left\|\sum_{j\in D} f_j\right\|.$$

□

With the objective to get the best possible constants in some results of this dissertation, we present the following result that improves the constant of item (ii) of the above corollary considering a sequence $(a_j)_{j\in J}$ with $0\le a_j\le 1$. This proposition can be found in [8].

Proposition 1.3.4. Let $\mathbb{X}$ be a Banach space, J a finite set, $0\le a_j\le 1$ for every $j\in J$ and $\{f_j\}_{j\in J}\subset\mathbb{X}$. Then,

$$\left\|\sum_{j\in J} a_j f_j\right\| \le \sup_{A\subseteq J}\left\|\sum_{j\in A} f_j\right\|.$$

Proof. By approximation, we can assume that $0\le a_j<1$ for all $j\in J$. Denote

$$\mathscr{A} = \{(\beta_k)_{k=1}^\infty \in \{0,1\}^{\mathbb{N}} \colon |\{k\in\mathbb{N}\colon \beta_k=0\}| = \infty\}.$$

Note that for each $0\le a<1$ there is a unique $(\beta_k)_{k=1}^\infty\in\mathscr{A}$ such that $a=\sum_{k=1}^\infty \beta_k 2^{-k}$. Put

$$a_j = \sum_{k=1}^\infty \beta_{j,k}2^{-k}, \quad (\beta_{j,k})_{k=1}^\infty \in \mathscr{A}$$

and denote

$$A_k = \{j\in J\colon \beta_{j,k}=1\}.$$

Then,

$$\begin{aligned}
\left\|\sum_{j\in J} a_j f_j\right\| &= \left\|\sum_{j\in J}\left(\sum_{k=1}^\infty \beta_{j,k}2^{-k} f_j\right)\right\| \\
&= \left\|\sum_{k=1}^\infty 2^{-k}\left(\sum_{j\in J}\beta_{j,k} f_j\right)\right\| \\
&= \left\|\sum_{k=1}^\infty 2^{-k}\left(\sum_{j\in A_k} f_j\right)\right\| \\
&\le \left(\sum_{k=1}^\infty 2^{-k}\right)\sup_{k\in\mathbb{N}}\left\|\sum_{j\in A_k} f_j\right\|.
\end{aligned}$$

Since $\sum_{k=1}^\infty 2^{-k}=1$ we are done. □

Chapter 2

Thresholding Greedy Algorithm and Greedy-type bases in Banach spaces

In this chapter we introduce and study the *Thresholding Greedy Algorithm* and its efficiency with respect to bases in Banach spaces. The most important classes of bases, in relation with this algorithm, are the following: greedy, almost-greedy, partially-greedy and quasi-greedy bases (see [60], [32] and [84], respectively). Many results stated in this chapter are known in the literature, but in some cases we have decided to include the proofs when these can be simplified or slightly improved with the respect to the original presentations.

The notion of greedy operators $(G_m)_{m=1}^{\infty}$ was first introduced by S. V. Konyagin and V. N. Temlyakov in [60]. More precisely, for an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$, $x \in \mathbb{X}$ and $m \in \mathbb{N}$, these are defined by

$$G_m[\mathscr{B},\mathbb{X}](x) := G_m(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

where A is any set so that $|A| = m$ and

$$\min_{n\in A} |\mathbf{e}_n^*(x)| \geq \max_{n\notin A} |\mathbf{e}_n^*(x)|.$$

In this case, $G_m(x)$ and the set $A = \operatorname{supp}(G_m(x))$ are called a *greedy sum* and a *greedy set* of x of order m, respectively.

The most natural way to describe a greedy sum of an element $x \in \mathbb{X}$ is to take an injective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{supp}(x) \subseteq \pi(\mathbb{N})$ and $(|\mathbf{e}_{\pi(j)}^*(x)|)_{j=1}^{\infty}$ is non-increasing and then, consider the partial sums

$$G_m(x) = \sum_{j=1}^{m} \mathbf{e}_{\pi(j)}^*(x)\mathbf{e}_{\pi(j)}.$$

Every such map π is call a *greedy ordering*. Note that a greedy ordering of a vector x needs not to be unique because we can have coefficients with the same modulus. It is then natural to introduce a specific "natural" ordering. Define the *natural greedy ordering* of $x \in \mathbb{X}$ as the mapping $\rho : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{supp}(x) \subseteq \rho(\mathbb{N})$ and such that if $j < k$, then either $|\mathbf{e}_{\rho(j)}^*(x)| > |\mathbf{e}_{\rho(k)}^*(x)|$ or $|\mathbf{e}_{\rho(j)}^*(x)| = |\mathbf{e}_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$.

The *m-th greedy sum* of x is given by this natural order, that is,

$$\mathscr{G}_m[\mathscr{B},\mathbb{X}](x) := \mathscr{G}_m(x) = \sum_{n=1}^{m} \mathbf{e}_{\rho(n)}^*(x)\mathbf{e}_{\rho(n)},$$

and it is uniquely determined since if $x \in \mathbb{X}$, for every $m \in \mathbb{N}$ there is a unique set $A_m(x) \subseteq \mathbb{N}$ such that $A_m(x) = \{\rho(1), ..., \rho(m)\}$ and $A_m(x)$ is called the *m-th greedy set*. The sequence of maps $(\mathscr{G}_m)_{m=1}^\infty$ is known as the *Thresholding Greedy Algorithm* (TGA) associated to $\mathscr{B}$ in $\mathbb{X}$.

Remark 2.0.1 ([85])**.** The operator $\mathscr{G}_m$ is neither linear nor continuous. For instance, if we define the elements:

$$x_n := \frac{n^2+1}{n^2}\mathbf{1}_{J_1} + \mathbf{1}_{J_2},$$

and

$$y_n := \mathbf{1}_{J_1} + \frac{n^2+1}{n^2}\mathbf{1}_{J_2},$$

where J_1 and J_2 are disjoint and $|J_i| = m$ for $i = 1, 2$, then, both elements converge to $\mathbf{1}_{J_1 \cup J_2}$ but,

$$\mathscr{G}_m(x_n) = \frac{n^2+1}{n^2}\mathbf{1}_{J_1}, \ \mathscr{G}_m(y_n) = \frac{n^2+1}{n^2}\mathbf{1}_{J_2},$$

hence, $\mathscr{G}_m(x_n) \to \mathbf{1}_{J_1}$ and $\mathscr{G}_m(y_n) \to \mathbf{1}_{J_2}$. Thus, $\mathscr{G}_m$ is not continuous. To see that the operator is not linear, we can take the following elements:

$$x = \sum_{i=1}^{3} \mathbf{e}_i + \sum_{j=4}^{\infty} \frac{1}{j^2}\mathbf{e}_j, \ y = -\sum_{i=1}^{3} \mathbf{e}_i + \sum_{j=4}^{\infty} \frac{1}{j^2}\mathbf{e}_j,$$

where $(\mathbf{e}_i)_{i=1}^\infty$ is an SM-basis. Then, $\mathscr{G}_3(x) = \sum_{i=1}^3 \mathbf{e}_i$ and $\mathscr{G}_3(y) = -\sum_{i=1}^3 \mathbf{e}_i$, but $\mathscr{G}_3(x+y) = \frac{\mathbf{e}_4}{4^2} + \frac{\mathbf{e}_5}{5^2} + \frac{\mathbf{e}_6}{6^2}$.

Remark 2.0.2 ([10, pg. 262])**.** The adjective "Thresholding" of the TGA has the following explanation. If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis, for each $\varepsilon > 0$, the *Thresholding operator* is defined as

$$T_\varepsilon[\mathscr{B}, \mathbb{X}](x) := T_\varepsilon(x) = \sum_{\{n : |\mathbf{e}_n^*(x)| > \varepsilon\}} \mathbf{e}_n^*(x)\mathbf{e}_n.$$

Since $\lim_n \mathbf{e}_n^*(x) = 0$ (see Lemma 1.1.3), then $T_\varepsilon(x)$ is well defined for each $x \in \mathbb{X}$. Define now

$$\mathbb{N}_x = \{m \in \mathbb{N} : \text{ there is a strictly greedy sum of } x \text{ of order } m\},$$

where a *strictly greedy sum* of x with order m is a greedy sum $G_m(x)$ satisfying that

$$\min_{n \in \operatorname{supp}(G_m(x))} |\mathbf{e}_n^*(x)| > \max_{n \notin \operatorname{supp}(G_m(x))} |\mathbf{e}_n^*(x)|.$$

Hence, for $\varepsilon > 0$ and $m \in \mathbb{N}_x$, we have

$$T_\varepsilon(x) = G_{m_\varepsilon}(x) \text{ and } G_m(x) = T_{\varepsilon_m}(x),$$

where

$$m_\varepsilon = |\{n \in \mathbb{N} : |\mathbf{e}_n^*(x)| > \varepsilon\}|, \quad \min_{n \in \operatorname{supp}(G_m(x))} |\mathbf{e}_n^*(x)| > \varepsilon_m \geq \max_{n \notin \operatorname{supp}(G_m(x))} |\mathbf{e}_n^*(x)|.$$

In some parts of this chapter, we will use the following lemma that connects the greedy sets of an element $x \in \mathbb{X}$, with the greedy sets of an element $y \in \mathbb{X}$ where $|\operatorname{supp}(y)| < \infty$ and

$\|x-y\| < \varepsilon$ for any $\varepsilon > 0$.

Lemma 2.0.3 ([66, Lemma 2.2])**.** Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$. Then, if $x \in \mathbb{X}$ and $A_m(x)$ is the m-th greedy set of x, for any $\varepsilon > 0$, there exists $y \in \mathbb{X}$ with finite support such that $\|x-y\| < \varepsilon$ and $A_m(x) = A_m(y)$.

2.1 Quasi-greedy bases

For an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ we want to study if the series $\sum_{n\geq 1} \mathbf{e}^*_{\rho(n)}(x)\mathbf{e}_{\rho(n)}$ converges to x for the natural greedy ordering ρ and for all $x \in \mathbb{X}$. Of course, if our basis is unconditional, the series converges, but we study a weaker notion than unconditionality that guarantees the convergence of the TGA.

Definition 2.1.1 ([60])**.** An SM-basis $\mathscr{B}$ in a Banach space is *quasi-greedy* if there exists a positive constant C such that

$$\|x - \mathscr{G}_m(x)\| \leq C\|x\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{2.1}$$

The least constant C in (2.1) is denoted by $C_q[\mathscr{B}, \mathbb{X}] := C_q$ and we will say that $\mathscr{B}$ is C_q*-quasi-greedy*.

Thanks to the following lemma, we may use in (2.1) $\mathscr{G}_m(x)$ or any other greedy sum $G_m(x)$.

Lemma 2.1.2 ([10, Lemma 10.2.6])**.** Let $\mathscr{B}$ be an SM-basis in a Banach space $\mathbb{X}$. The following are equivalent:

- $\mathscr{B}$ is C_q-quasi-greedy.
- There exists a constant C such that for all $x \in \mathbb{X}$, all $m \in \mathbb{N}$, and all greedy sums $G_m(x)$,

$$\|x - G_m(x)\| \leq C\|x\|.$$

- For every $m \in \mathbb{N}$ and $x \in \mathbb{X}$, there exist a greedy sum $G_m(x)$ such that

$$\|x - G_m(x)\| \leq C\|x\|,$$

  where C is an absolute constant.

The least constant C in any of the above estimates is also C_q.

The relation between the uniform boundedness of the *m-th greedy error* $\|x - \mathscr{G}_m(x)\|$ and the convergence of the TGA was given by P. Wojtaszczyk in [84].

Theorem 2.1.3 ([84])**.** $\mathscr{B}$ is quasi-greedy if and only if $\sum_{n\geq 1} \mathbf{e}^*_{\rho(n)}(x)\mathbf{e}_{\rho(n)}$ converges to x for every $x \in \mathbb{X}$.

The above theorem was proved by P. Wojtaszczyk but we try to give a more complete proof here, which also clarifies the role of the following condition. We say that an SM-basis is *strong* (see [73, Definition 8.4]) if

$$[\mathbf{e}_n : n \in A] = \{x \in \mathbb{X} : \mathbf{e}^*_n(x) = 0 \ \text{ if } n \notin A\}.$$

The reason to use this condition is the following: if $\mathscr{B}$ is an SM-basis and not strong, then, there exists some set $A \subset \mathbb{N}$ (necessarily infinite) and some $x_0 \in \mathbb{X}$ with $\operatorname{supp}(x_0) \subset A$ such that

$$\delta = \operatorname{dist}(x_0, [\mathbf{e}_n]_{n\in A}) > 0.$$

Since $\operatorname{supp}(\mathscr{G}_m(x_0))$ is always a subset of A, the algorithm can not converge. Of course, if the TGA converges, the basis is necessarily strong.

We present a complete proof of Wojtaszczyk's theorem and show in addition that every SM-quasi-greedy basis is necessarily strong (see [8]).

Theorem 2.1.4. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space $\mathbb{X}$. The following conditions are equivalent:

(i) $\mathscr{B}$ is quasi-greedy.

(ii) For every $x \in \mathbb{X}$ the greedy series of x converges.

(iii) For every $x \in \mathbb{X}$ the greedy algorithm $(\mathscr{G}_m(x))_{m=1}^{\infty}$ is bounded.

Corollary 2.1.5. Every SM-quasi-greedy basis is strong.

Prior to do it, we establish a couple of lemmas that we will need.

Lemma 2.1.6. Let $\mathscr{B}$ be an SM-basis in a Banach space $\mathbb{X}$. Assume that $\mathscr{B}$ is C_q-quasi-greedy and let $x, z \in \mathbb{X}$ and D be a greedy set of $x - z$ such that $\operatorname{supp}(z) \subseteq D$. Then $\|x - P_D(x)\| \leq C_q\|x - z\|$.

Proof. Since $z = P_D(z)$, we have

$$\|x - P_D(x)\| = \|x - z - P_D(x - z)\| \leq C_q\|x - z\|. \qquad \square$$

Lemma 2.1.7. If an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ of a Banach space $\mathbb{X}$ is not quasi-greedy, then for every constant $R > 0$ and every finite set $A \subseteq \mathbb{N}$ there exists $x \in \mathbb{X}$ with finite support disjoint with A and a strictly greedy set B of x such that $\|P_B(x)\| > R\|x\|$.

Proof. Pick $R_0 = (M + R + MR)$, where $M = \max_{D\subseteq A} \|P_D\|$. Let us fix a vector $g_0 \in \mathbb{X}$ and a greedy set B_0 of g_0 such that $\|P_{B_0}(g_0)\| > R_0\|g_0\|$. If we put $g_1 = g_0 - P_A(g_0)$ then $\|g_1\| \leq (1 + M)\|g_0\|$. Let $B = B_0 \setminus A$ and $D = B_0 \cap A$, so that B is a greedy set of g_1. We have

$$\begin{aligned}
\|P_B(g_1)\| &= \|P_B(g_0)\| \\
&= \|P_{B_0}(g_0) - P_D(g_0)\| \\
&\geq \|P_{B_0}(g_0)\| - \|P_D(g_0)\| \\
&> (R_0 - M)\|g_0\| \\
&\geq \frac{R_0 - M}{1 + M}\|g_1\| \\
&= R\|g_1\|.
\end{aligned}$$

For each $t > 0$, using Lemma 2.0.3, we pick $x_t \in \mathbb{X}$ finitely supported such that

$$\left\| x_t - g_1 - t \sum_{n\in B} \operatorname{sign}(\mathbf{e}_n^*(g_1))\, \mathbf{e}_n \right\| < \frac{t}{2\mathbf{c}^*},$$

B a strictly greedy set of x_t and that $\lim_{t\to 0^+} x_t = g_1$. Then $\lim_{t\to 0^+} P_B(x_t) = P_B(g_1)$. Consequently, for t small enough, $\|P_B(x_t)\| > R\|x_t\|$. □

Proof of Theorem 2.1.4. We start by proving that (i) implies (ii). Let $x \in \mathbb{X}$ and $\varepsilon > 0$. We pick $z = \sum_{n\in B} a_n \mathbf{e}_n$ with B finite such that $\|x-z\| \le \varepsilon/C_{qg}^2$. Perturbing the vector z if necessary we can assume that B is nonempty and that $a_n \neq \mathbf{e}_n^*(x)$, i.e., $\mathbf{e}_n^*(x-z) \neq 0$, for every $n \in B$. Set $\nu = \min_{n\in B} |\mathbf{e}_n^*(x-z)| > 0$. We have

$$\operatorname{supp}(z) \subseteq B \subseteq D := \{n \in \mathbb{N} \colon |\mathbf{e}_n^*(x-z)| \ge \nu\}.$$

Since D is a strictly greedy set of $x-z$, applying Lemma 2.1.6 we obtain,

$$\|x - P_D(x)\| \le C_q\|x-z\| \le \frac{\varepsilon}{C_q}.$$

Set $\mu = \min_{n\in D\cap\operatorname{supp}(x)} |\mathbf{e}_n^*(x)|$ (with the convention that $\mu = \infty$ if $D\cap\operatorname{supp}(x) = \emptyset$) and let $m \ge |\{n \in \mathbb{N} \colon |\mathbf{e}_n^*(x)| \ge \mu\}|$. Since $D\cap\operatorname{supp}(x) \subseteq A_m(x)$, the set $G := A_m(x) \setminus (D\cap\operatorname{supp}(x))$ is greedy for $g := x - P_D(x)$. Consequently,

$$\left\| x - \sum_{n\in A_m(x)} \mathbf{e}_n^*(x)\,\mathbf{e}_n \right\| = \|g - P_G(g)\| \le C_q\|g\| \le \varepsilon.$$

As it is obvious that (ii) implies (iii), we close the proof by proving that that if (i) does not hold then (iii) does not hold either. Under the assumption that $\mathscr{B}$ is not quasi-greedy we recursively construct a sequence $(x_k)_{k=1}^\infty$ in $\mathbb{X}$ and a sequence $(B_k)_{k=1}^\infty$ of finite subsets of $\mathbb{N}$ such that, if $\mu_1 = \infty$ and

$$\mu_k = \min\{|\mathbf{e}_n^*(x_{k-1})| \colon n \in \operatorname{supp}(x_{k-1})\}$$

for $k \ge 2$, and satisfying

(a) $A_k := \operatorname{supp}(x_k)$ is finite and disjoint with $\cup_{i=1}^{k-1}\operatorname{supp}(x_i)$,

(b) B_k is a strictly greedy set of x_k,

(c) $\|x_k\| \le 2^{-k}$,

(d) $\|P_{B_k}(x_k)\| > 2^k$, and

(e) $\max\{|\mathbf{e}_n^*(x_k)| : n \in \mathbb{N}\} < \mu_k$

for every $k \in \mathbb{N}$. To see that this is possible, suppose we have manufactured x_i and B_i for $i < k$. Put

$$\gamma_k = \min\{2^{-2k}, (2\mathbf{c}^*)^{-1}2^{-k}\mu_k\}.$$

By Lemma 2.1.7, there exists $x_k \in \mathbb{X}$ whose support is finite and disjoint with $\cup_{i=1}^{k-1}\operatorname{supp}(x_i)$ and a strictly greedy set B_k of x_k such that $\|P_{B_k}(x_k)\| > \gamma_k^{-1}\|x_k\|$. By homogeneity we can choose x_k satisfying $\|x_k\| = 2^k\gamma_k$, so that (d) holds. Since $\gamma_k \le 2^{-2k}$ (c) also holds. For any $n \in \mathbb{N}$ we have

$$|\mathbf{e}_n^*(x_k)| \le \|\mathbf{e}_n^*\|\,\|x_k\| \le \|\mathbf{e}_n^*\|\frac{\mu_k}{2\mathbf{c}^*} \le \frac{\mu_k}{2},$$

and so (e) also holds.

Since $\sum_{k=1}^{\infty}\|x_k\|<\infty$, the series $\sum_{k=1}^{\infty}x_k$ converges to some $x\in\mathbb{X}$ which satisfies

$$\mathbf{e}_n^*(x)=\begin{cases}\mathbf{e}_n^*(x_k) & \text{if } n\in A_k,\\ 0 & \text{if } n\notin\cup_{k=1}^{\infty}A_k.\end{cases}$$

By (b) and (e) both $D_k=\cup_{i=1}^{k-1}A_i$ and $F_k=D_k\cup B_k$ are strictly greedy sets of x. Thus, if $m_k=|D_k|$ and $q_k=|F_k|$,

$$\|\mathscr{G}_{m_k+q_k}(x)-\mathscr{G}_{m_k}(x)\|=\|P_{F_k}(x)-P_{D_k}(x)\|=\|P_{B_k}(x_k)\|>2^k.$$

for every $k\in\mathbb{N}$. We infer that $\sup_m\|\mathscr{G}_m(x)\|=\infty$. □

Example 2.1.8 (Example of a non quasi-greedy basis)**.** Let $\mathbb{X}$ be the closure of the set of all finite sequences $\mathbf{a}=(a_n)_{n\in\mathbb{N}}\in c_{00}$ with the norm

$$\|\mathbf{a}\|=\sup_{m\geq 1}\left|\sum_{n=1}^{m}a_n\right|<\infty.$$

The canonical basis $\mathscr{B}=(\mathbf{e}_n)_{n\in\mathbb{N}}$ is Schauder. To show that $\mathscr{B}$ is not quasi-greedy, we take the element $\mathbf{a}=(-1,2+\varepsilon,-2,...,2+\varepsilon,-2,0,...)$, where $(2+\varepsilon,-2)$ appears m times with $\varepsilon>0$. The norm is $\|\mathbf{a}\|=1+m\varepsilon$ and $\mathscr{G}_m(\mathbf{a})=(0,2+\varepsilon,0,...,2+\varepsilon,0,...)$ with $\|\mathbf{a}-\mathscr{G}_m(\mathbf{a})\|=1+2m$. Taking limits when ε goes to 0, $\sup_m\frac{\|\mathbf{a}-\mathscr{G}_m(\mathbf{a})\|}{\|\mathbf{a}\|}=\infty$.

Now, we present an example of a conditional quasi-greedy basis.

Example 2.1.9 ([60])**.** In c_{00}, define the norm of a sequence $\mathbf{a}=(a_n)_{n\in\mathbb{N}}$ by the formula

$$\|\mathbf{a}\|=\max\left\{\|\mathbf{a}\|_2,\ \sup_m\left|\sum_{n=1}^{m}\frac{a_n}{\sqrt{n}}\right|\right\},$$

where $\|\mathbf{a}\|_2$ is the norm in the space ℓ_2. Let $\mathbb{X}$ be the completion of c_{00} in c_0 under this norm and let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^{\infty}$ the canonical basis in $\mathbb{X}$. S. V. Konyagin and V. N. Temlyakov proved that the basis is quasi-greedy. Also, F. Albiac and N. J. Kalton showed ([10, Example 10.2.9]) that the quasi-greedy constant C_q is upper bounded by $3+\sqrt{2}$. To show that the basis is conditional, taking the element $\mathbf{a}_m=\sum_{n=1}^{m}1/\sqrt{n}$,

$$\|\mathbf{a}_m\|=\max\left\{\left(\sum_{n=1}^{m}\frac{1}{n}\right)^{1/2},\sum_{n=1}^{m}\frac{1}{n}\right\}\approx\ln(m).$$

On the other hand, taking the signs $\varepsilon_n=(-1)^n$,

$$\left\|\sum_{n=1}^{m}\frac{\varepsilon_n}{\sqrt{n}}\right\|=\max\left\{\left(\sum_{n=1}^{m}\frac{1}{n}\right)^{1/2},\sum_{n=1}^{m}\frac{\varepsilon_n}{n}\right\}\approx\sqrt{\ln(m)}.$$

Hence, the basis is conditional.

This last example shows that quasi-greediness is not equivalent to unconditionality, but, despite this fact, quasi-greediness implies some properties that unconditionality preserves, as for example the inequalities of Proposition 1.1.5. In the rest of this section, we show this result using the following definitions and some other results based on the ideas of [18].

Definition 2.1.10 ([84])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *unconditional for constant coefficients* if there exists a positive constant C such that for every finite set A,

$$\|\mathbf{1}_{\varepsilon B}\| \leq C\|\mathbf{1}_{\varepsilon A}\|,\ \forall B \subseteq A, \forall \varepsilon \in \Psi_A. \tag{2.2}$$

The least constant C in (2.2) is denoted by $K_c[\mathscr{B},\mathbb{X}] := K_c$ and we will say that $\mathscr{B}$ is K_c-*unconditional for constant coefficients*.

Of course, if a basis $\mathscr{B}$ is C_q-quasi-greedy, then $\mathscr{B}$ is K_c-unconditional for constant coefficients with $K_c \leq C_q$.

Proposition 2.1.11. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$. The following statements are equivalent:

a) $\mathscr{B}$ is unconditional for constant coefficients.

b) There exists a positive constant C_1 such that for any $\varepsilon, \eta \in \Psi_A$ with A a finite set,

$$\|\mathbf{1}_{\varepsilon A}\| \leq C_1\|\mathbf{1}_{\eta A}\|. \tag{2.3}$$

c) There is a constant C_2 such that for any finite $A \subset \mathbb{N}$ and $\varepsilon \in \Psi_A$,

$$\left\|\sum_{n\in A} a_n \mathbf{e}_n\right\| \leq C_2 \max_{n\in A} |a_n| \|\mathbf{1}_{\varepsilon A}\|,$$

whenever $a_n \in \mathbb{F}$, $n \in A$.

Also, if $\mathscr{B}$ is K_c-unconditional for constant coefficients, then $C_2 \leq C_1 \leq 2\kappa K_c$, where $\kappa = 1$ or 2 if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ respectively.

Proof. First, we show that a) $\Rightarrow$ b). The proof for the real case can be found in [32]. Here, we consider also the complex case. Take a finite set A and $\varepsilon, \eta \in \Psi_A$. One the one hand, taking into account that

$$\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \frac{\varepsilon_n}{\eta_n} \eta_n \mathbf{e}_n,$$

using (ii) of Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D\subseteq A} \|\mathbf{1}_{\eta D}\|. \tag{2.4}$$

Now, using that the basis is unconditional for constant coefficients, $\|\mathbf{1}_{\eta D}\| \leq K_c\|\mathbf{1}_{\eta A}\|$. By (2.4),

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D\subseteq A} \|\mathbf{1}_{\eta D}\| \leq 2\kappa K_c\|\mathbf{1}_{\eta A}\|.$$

Prove now that b)$\Rightarrow$ c). If we take $t := \max_{j\in A}|a_j|$, by (i) of Corollary (1.3.3),

$$\left\|\sum_{n\in A} a_n\mathbf{e}_n\right\| \leq t \sup_{\eta\in\Psi_A}\|\mathbf{1}_{\eta A}\|. \tag{2.5}$$

Now, applying b) in (2.5), if $\varepsilon\in\Psi_A$,

$$\left\|\sum_{n\in A} a_n\mathbf{e}_n\right\| \leq C_1 \max_{n\in A}|a_n|\|\mathbf{1}_{\varepsilon A}\|.$$

The implication c) $\Rightarrow$ a) is trivial since if $\varepsilon\in\Psi_A$, using $a_n=\varepsilon_n$ if $n\in B$ and $a_n=0$ if $n\in A\setminus B$,

$$\|\mathbf{1}_{\varepsilon B}\| \leq C_2\|\mathbf{1}_{\varepsilon A}\|.$$

□

In Proposition 2.1.11, we have showed that quasi-greediness implies the right hand inequality of Proposition 1.1.5. To show the left hand inequality, we need the following result.

Proposition 2.1.12 ([32, Lemma 2.2])**.** Let $\mathscr{B}$ be an SM-basis in a real Banach space $\mathbb{X}$. If $\mathscr{B}$ is C_q-quasi-greedy or K_{su}-unconditional, then,

$$\min_{i\in\Lambda}|\mathbf{e}_i^*(x)|\|\mathbf{1}_{\varepsilon\Lambda}\| \leq \min\{2C_q, K_{su}\}\|x\|, \tag{2.6}$$

for every $x\in\mathbb{X}$ and Λ greedy set of x with $\varepsilon\equiv\{\operatorname{sign}(\mathbf{e}_n^*(x))\}$.

This result was proved by S. J. Dilworth et al. ([32]) using an Abel summation argument and for the real case. For the complex case, we present below a proof using the truncation operator that was introduced in [31]. For $z\in\mathbb{C}$, we define the *α-truncation* as

$$T_\alpha(z) = \alpha\operatorname{sign}(z) \text{ if } |z|\geq\alpha, \quad \text{and} \quad T_\alpha(z) = z \text{ if } |z|\leq\alpha.$$

We extend T_α to an operator in $\mathbb{X}$ by $T_\alpha(x)\sim\sum_{i=1}^\infty T_\alpha(\mathbf{e}_i^*(x))\mathbf{e}_i$, that is, if $x\in\mathbb{X}$,

$$T_\alpha(x) := \alpha\mathbf{1}_{\varepsilon\Lambda_\alpha} + (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_\alpha})(x), \tag{2.7}$$

where $\Lambda_\alpha = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$ and $\varepsilon\equiv\{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. Since Λ_α is a finite set, the operator T_α is well-defined for all $x\in\mathbb{X}$.

[31, Proposition 3.1] shows that the truncation operator is uniformly bounded when $\mathscr{B}$ is a C_q-quasi-greedy basis with $\|T_\alpha\|\leq 1+3C_q$. We have a similar proof in [18, Lemma 2.5] improving the boundedness constant.

Lemma 2.1.13. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. If $\mathscr{B}$ is a C_q-quasi-greedy, for all $\alpha>0$ and $x\in\mathbb{X}$ we have

$$\|T_\alpha(x)\| \leq C_q\|x\|, \quad \|(\mathrm{Id}_{\mathbb{X}} - T_\alpha)(x)\| \leq (1+C_q)\|x\|, \tag{2.8}$$

Moreover, if $\mathscr{B}$ is K_{su}-unconditional, for every finite set $A\subset\mathbb{N}$,

$$\|T_\alpha(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\| \leq K_{su}\|x\|. \tag{2.9}$$

Proof. Set $\alpha \in \mathbb{F}$. Notice first that

$$T_\alpha(x) = \int_0^1 \Big[\sum_n \chi_{[0,\frac{\alpha}{|\mathbf{e}_n^*(x)|}]}(s)\,\mathbf{e}_n^*(x)\mathbf{e}_n\Big]\,ds = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_{\alpha,s}})(x)\,ds, \tag{2.10}$$

where $\Lambda_{\alpha,s} = \{n : |\mathbf{e}_n^*(x)| > \frac{\alpha}{s}\}$ for each $s \in (0,1]$. The result follows from Minkowski's integral inequality applied to (2.10) and using the following two formulas derived from it:

$$(\mathrm{Id}_{\mathbb{X}} - T_\alpha)(x) = \int_0^1 P_{\Lambda_{\alpha,s}}(x)\,ds,$$

and

$$T_\alpha(\mathrm{Id}_{\mathbb{X}} - P_A)(x) = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_{\alpha,s}})(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\,ds = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{A\cup\Lambda_{\alpha,s}})(x)\,ds.$$

□

Lemma 2.1.14. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. If $\mathscr{B}$ is a C_q-quasi-greedy (or K_{su}-unconditional) and $x \in \mathbb{X}$ with $\varepsilon \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$, then, for all greedy set Λ of x,

$$\min_\Lambda |\mathbf{e}_n^*(x)|\, \|\mathbf{1}_{\varepsilon\Lambda}\| \le \min\{2C_q, K_{su}\}\, \|x\|. \tag{2.11}$$

Proof. Take $\alpha = \min_{n\in\Lambda} |\mathbf{e}_n^*(x)|$. Using (2.7) and (2.10),

$$\alpha \mathbf{1}_{\varepsilon\Lambda} = T_\alpha(x) - P_{\Lambda^c}(x) = \int_0^1 (P_\Lambda(x) - P_{\Lambda_{\alpha,s}}(x))\,ds.$$

Note that $\Lambda_{\alpha,s}$ is a greedy set of x and $\Lambda_{\alpha,s} \subseteq \Lambda_\alpha \subset \Lambda$. Hence

$$\|P_\Lambda(x) - P_{\Lambda_{\alpha,s}}(x)\| \le \min\{2C_q, K_{su}\}\|x\|, \quad 0 < s \le 1.$$

The result now follows. □

Corollary 2.1.15. If $\mathscr{B}$ is an SM-quasi-greedy basis, for all finite set A,

$$\min_{n\in A} |a_n| \|\mathbf{1}_A\| \le 4\kappa C_q^2 \left\| \sum_{n\in A} a_n \mathbf{e}_n \right\|, \ \forall (a_n)_{n\in A} \subset \mathbb{F}.$$

Proof. Using Lemma 2.1.14 and (2.3) the result follows. □

Remark 2.1.16. By Proposition 2.1.11 and Corollary 2.1.15, we have that for all finite set A and $\forall (a_n)_{n\in A} \subset \mathbb{F}$,

$$C_1 \min_{n\in A} |a_n| \|\mathbf{1}_A\| \le \left\| \sum_{n\in A} a_n \mathbf{e}_n \right\| \le C_2 \max_{n\in A} |a_n| \|\mathbf{1}_A\|, \tag{2.12}$$

for some constants $C_1 = C_1(C_q, \kappa)$ and $C_2 = C_2(C_q, \kappa)$, where C_q is the quasi-greedy constant and κ is defined as in Proposition 2.1.11. As we have commented, this consequence exhibits that quasi-greedy bases are rather close to unconditional bases in the sense of Proposition 1.1.5.

2.2 Democracy-like properties

In the following sections, to characterize greedy-type bases, we will need some extensions of the concept of "democracy", as defined originally by S. V. Konyagin and V. N. Temlyakov in [60]. We present now the definitions and we study some properties and relations between them. First, we introduce the following notation:

$$\mathbb{N}^m = \{A \subset \mathbb{N} : |A| = m\}, \quad \mathbb{N}^{<\infty} = \cup_{m=0}^{\infty}\mathbb{N}^m,$$

and

$$\max A = \max_{j\in A} j, \quad \min A = \min_{j\in A} j, \quad A < B \text{ if } \max A < \min B.$$

Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space $\mathbb{X}$ and consider the following condition:

$$\|x + \mathbf{1}_{\varepsilon A}\| \leq C\|x + \mathbf{1}_{\varepsilon' B}\|, \tag{2.13}$$

which involves a positive constant C, two finite sets $A, B \in \mathbb{N}^{<\infty}$ with $|A| \leq |B|$, a vector $x \in \mathbb{X}$ such that $|x|_\infty \leq 1$, where we remind that $|x|_\infty := \sup_n |\mathbf{e}_n^*(x)|$, and $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$.

Definition 2.2.1. We say that $\mathscr{B}$ is *democratic* if there exists a positive constant C such that (2.13) holds for every sets $A, B \in \mathbb{N}^{<\infty}$ with $|A| \leq |B|$, $x = 0$ and $\varepsilon \equiv \varepsilon' \equiv 1$. The least constant that verifies (2.13) under these conditions is denoted by $C_d[\mathscr{B}, \mathbb{X}] := C_d$ and we will say that $\mathscr{B}$ is *C_d-democratic*.

Definition 2.2.2. We say that $\mathscr{B}$ is *super-democratic* if there exists a positive constant C such that (2.13) holds for every $A, B \in \mathbb{N}^{<\infty}$ with $|A| \leq |B|$, for every choice of $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$ and for $x = 0$. The least constant that verifies (2.13) under these conditions is denoted by $C_s[\mathscr{B}, \mathbb{X}] := C_s$ and we will say that $\mathscr{B}$ is *C_s-super-democratic*. If in addition, $A \cap B = \emptyset$, we will say that $\mathscr{B}$ is *C_{sd}-disjoint-super-democratic*.

Definition 2.2.3. We say that $\mathscr{B}$ is *super-conservative* if there exists a positive constant C such that (2.13) holds for every $A, B \in \mathbb{N}^{<\infty}$ with $|A| \leq |B|$ and $A < B$, for every choice of $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$ and for $x = 0$. The least constant that verifies (2.13) under these conditions is denoted by $C_{sc}[\mathscr{B}, \mathbb{X}] := C_{sc}$ and we will say that $\mathscr{B}$ is *C_c-super-conservative*. If in addition, $\varepsilon \equiv \varepsilon' \equiv 1$, we will say that $\mathscr{B}$ is *C_c-conservative*.

Definition 2.2.4. We say that $\mathscr{B}$ is *symmetric for largest coefficients* (SLC for short) if there exists a positive constant C such that (2.13) holds for every pair of sets A and B such that $A, B \in \mathbb{N}^{<\infty}$, $|A| \leq |B|$, $A \cap B = \emptyset$, for every $x \in \mathbb{X}$ with $|x|_\infty \leq 1$ and $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$, and for every choice of $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$. The least constant that verifies (2.13) under these conditions is denoted by $C_a[\mathscr{B}, \mathbb{X}] := C_a$ and we will say that $\mathscr{B}$ is *C_a-symmetric for largest coefficients*.

We define the set $\mathfrak{F}$ as the family of all 5-tuples $(A, B, \varepsilon, \varepsilon', x)$ such that

$$A, B \in \mathbb{N}^{<\infty}, \quad |A| \leq |B|, \quad |x|_\infty \leq 1, \quad \operatorname{supp}(x) \cap (A \cup B) = \emptyset, \quad \varepsilon \in \Psi_A, \quad \varepsilon' \in \Psi_B, \tag{2.14}$$

and consider the following subsets:

- $\mathfrak{F}_d$ is the subset of $\mathfrak{F}$ where $A \cap B = \emptyset$.

- $\mathfrak{F}_c$ is the subset of $\mathfrak{F}_d$ where $A < B$.

Finally, we write $\mathfrak{F}'$ for the subset of $\mathfrak{F}$ where $x = 0$ and likewise $\mathfrak{F}'_d$ and $\mathfrak{F}'_c$. The next table collects all the previous definitions.

Name	**Inequality**	
Democracy	$\|\mathbf{1}_A\| \le C_d \|\mathbf{1}_B\| \quad \forall (A,B,1,1) \in \mathfrak{F}'$	(2.15)
Super-democracy	$\|\mathbf{1}_{\varepsilon A}\| \le C_s \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'$	(2.16)
Disjoint-super-democracy	$\|\mathbf{1}_{\varepsilon A}\| \le C_{sd} \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'_d$	(2.17)
Conservative	$\|\mathbf{1}_A\| \le C_c \|\mathbf{1}_B\| \quad \forall (A,B,1,1) \in \mathfrak{F}'_c$	(2.18)
Super-conservative	$\|\mathbf{1}_{\varepsilon A}\| \le C_{sc} \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'_c$	(2.19)
SLC	$\|x+\mathbf{1}_{\varepsilon A}\| \le C_a \|x+\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d$	(2.20)

Hereunder, we present a result connecting the democracy-like properties that can be found, for instance, in [18].

Proposition 2.2.5. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_s-super-democratic, then $\mathscr{B}$ is disjoint-super-democratic, democratic, super-conservative, and conservative with

$$C_c \le C_d \le C_s \text{ and } C_c \le C_{sc} \le C_{sd} \le C_s.$$

b) If $\mathscr{B}$ is C_a-SLC, then $\mathscr{B}$ is disjoint-super-democratic and super-democratic with

$$C_{sd} \le C_a \text{ and } C_s \le (2\kappa) C_a,$$

where $\kappa = 1$ if $\mathbb{F} = \mathbb{R}$ and $\kappa = 2$ if $\mathbb{F} = \mathbb{C}$.

c) $\mathscr{B}$ is super-democratic if and only if $\mathscr{B}$ is unconditional for constant coefficients and democratic. Moreover,

$$\max\{C_d, K_c\} \le C_s \le 4\kappa^2 C_d K_c.$$

Proof. The proof of a) is trivial by the definitions. We prove now b). Of course, (2.20) implies (2.17) with $C_{sd} \leq C_a$. Now, take $(A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'$ and show that (2.20) implies (2.16) with $(2\kappa)^{-1}C_s \leq C_a$. Using item (ii) of Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D\subseteq A} \|\mathbf{1}_{\varepsilon' D}\|. \tag{2.21}$$

Take $D \subseteq A$. We have the following decomposition: $\mathbf{1}_{\varepsilon' D} = \mathbf{1}_{\varepsilon'(D\setminus B)} + \mathbf{1}_{\varepsilon'(D\cap B)}$. Since $|A| \leq |B|$, it is obvious that $|D\setminus B| \leq |B\setminus D|$. Applying then (2.20),

$$\|\mathbf{1}_{\varepsilon' D}\| = \|\mathbf{1}_{\varepsilon'(D\setminus B)} + \mathbf{1}_{\varepsilon'(D\cap B)}\| \leq C_a\|\mathbf{1}_{\varepsilon'(B\setminus D)} + \mathbf{1}_{\varepsilon'(B\cap D)}\| = \|\mathbf{1}_{\varepsilon' B}\|. \tag{2.22}$$

By (2.21) and (2.22), we obtain that $C_s \leq 2\kappa C_a$.

Finally, we prove c). Of course, if $\mathscr{B}$ is C_s-super-democratic, $\mathscr{B}$ is C_d-democratic and K_c-unconditional for constant coefficients with $\max\{K_c, C_d\} \leq C_s$. We prove the converse. Take $(A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'$ and prove (2.16). First, using (ii) of Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D\subseteq A} \|\mathbf{1}_{D}\|. \tag{2.23}$$

On the other hand, using democracy and Proposition 2.1.11, $\|\mathbf{1}_D\| \leq C_d\|\mathbf{1}_B\| \leq 2\kappa C_d K_c \|\mathbf{1}_{\varepsilon' B}\|$. Using the last one inequality in (2.23), we obtain the result. □

We now discuss some properties about the symmetry for largest coefficients.

First property: elements with finite support. The following lemma proved in [21] shows that in (2.20) it is enough to consider that $x \in \mathbb{X}$ has finite support.

Lemma 2.2.6. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$. Suppose D is a finite subset of $\mathbb{N}$, and $x \in \mathbb{X}\setminus\{0\}$ satisfies $\text{supp}(x) \cap D = \emptyset$. Then, for any $\varepsilon > 0$, there exists a finitely supported $y \in \mathbb{X}$, so that $\|x-y\| < \varepsilon$, $\text{supp}(y) \cap D = \emptyset$, and $\max_n |\mathbf{e}_n^*(x)| = \max_n |\mathbf{e}_n^*(y)|$.

Proof. By scaling, we can assume that $\max_n |\mathbf{e}_n^*(x)| = 1$ (then $\|x\| \geq 1/\mathbf{c}^*$). Clearly, $P_D(x) = 0$ and $P_{D^c}(x) = x$. Since $\mathscr{B}$ is complete, for every $\delta > 0$, there exists a finitely supported $z \in \mathbb{X}$ so that $\|x - z\| < \delta/\|P_{D^c}\|$. Let $u = P_{D^c}(z)$, then, $\|x-u\| = \|P_{D^c}(x-z)\| < \delta$. For every $n \in \mathbb{N}$, $|\mathbf{e}_n^*(x-u)| < \mathbf{c}^*\delta$, hence, $C = \max_n |\mathbf{e}_n^*(u)|$ is in $(1-\mathbf{c}^*\delta, 1+\mathbf{c}^*\delta)$. Now, let $y = u/C$. Then, $\max_n |\mathbf{e}_n^*(y)| = 1$ and

$$\|x-y\| \quad \leq \quad \|x-u\| + |1-C^{-1}|\|u\| < \delta + \frac{\mathbf{c}^*\delta}{1-\mathbf{c}^*\delta}(\|x\|+\delta).$$

Thus, picking δ sufficiently small we can achieve $\|x-y\| < \varepsilon$. □

Second property: an equivalence. We prove now an equivalence of the symmetry for largest coefficients. The following result is an extension of [4, Proposition 3.7] and [8, Proposition 6.3]. First, we need the following definition.

Definition 2.2.7. An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *quasi-greedy for largest coefficients* if there is a constant C such that

$$\|\mathbf{1}_{\varepsilon A}\| \leq C\|x + \mathbf{1}_{\varepsilon A}\|, \tag{2.24}$$

for any $x \in \mathbb{X}$ with $|x|_\infty \leq 1$, $A \in \mathbb{N}^{<\infty}$, $\operatorname{supp}(x) \cap A = \emptyset$ and $\varepsilon \in \Psi_A$. The least constant C in (2.24) is denoted by $C_{ql}[\mathscr{B},\mathbb{X}] := C_{ql}$ and we will say that $\mathscr{B}$ is *C_{ql}-quasi-greedy for largest coefficients*.

Proposition 2.2.8. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$. The following statements are equivalent:

a) $\mathscr{B}$ is C_a-SLC.

b) For any $\varepsilon' \in \Psi_B$, $x \in \mathbb{X}$ with $|x|_\infty \leq 1$, $B \cap \operatorname{supp}(x) = \emptyset$ and $|A| \leq |B|$, there exists a constant C_1 such that
$$\|x\| \leq C_1 \|x - P_A(x) + \mathbf{1}_{\varepsilon' B}\|. \tag{2.25}$$

c) $\mathscr{B}$ is C_{ql}-quasi-greedy for largest coefficients and C_{sd}-disjoint-super-democratic.

d) For every $(A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d$ with $|A| = |B|$ and $t \geq \sup_j |\mathbf{e}_j^*(x)|$, there exists a positive constant C_2 such that
$$\|x + t\mathbf{1}_{\varepsilon A}\| \leq C_2 \|x + t\mathbf{1}_{\varepsilon' B}\|. \tag{2.26}$$

Also, $C_a = C_1 = C_2$, $C_a \leq 1 + (1 + C_{sd})C_{ql}$ and $C_{ql} \leq 1 + C_a$.

Proof. First, we show the equivalence a) $\Leftrightarrow$ b). Assume that $\mathscr{B}$ is C_a-SLC. Take A, B, x, ε' as in the statement of b). From (2.20), for any $\varepsilon \in \Psi_A$,
$$\|P_{A^c}(x) + \mathbf{1}_{\varepsilon A}\| \leq C_a \|P_{A^c}(x) + \mathbf{1}_{\varepsilon' B}\|.$$

Now, using Corollary 1.3.2,
$$\|x\| = \|P_{A^c}(x) + P_A(x)\| \leq \sup_{\varepsilon \in \Psi_A} \|P_{A^c}(x) + \mathbf{1}_{\varepsilon A}\| \leq C_a \|x - P_A(x) + \mathbf{1}_{\varepsilon' B}\|.$$

Now, we prove the implication b) $\Rightarrow$ a). Assume that we have (2.25) and take $(A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d$ to show (2.20). If we define $x' := x + \mathbf{1}_{\varepsilon A}$ and we apply (2.25) to x', we obtain the implication.

We prove now a) $\Leftrightarrow$ c). Assume that $\mathscr{B}$ is C_a-SLC. Using item b) of Proposition 2.2.5, $\mathscr{B}$ is disjoint-super-democratic with $C_{sd} \leq C_a$. To show that $\mathscr{B}$ is quasi-greedy for largest coefficients, take x, A, ε as in the definition. Then,
$$\|\mathbf{1}_{\varepsilon A}\| \leq \|x + \mathbf{1}_{\varepsilon A}\| + \|x\| \leq \|x + \mathbf{1}_{\varepsilon A}\| + C_a \|x + \mathbf{1}_{\varepsilon A}\|.$$

Hence, $\mathscr{B}$ is quasi-greedy for largest coefficients with $C_{ql} \leq 1 + C_a$. Now, we prove a) using c). Take $(A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d$ and show (2.20):
$$\begin{aligned}
\|x + \mathbf{1}_{\varepsilon A}\| &\leq \|x + \mathbf{1}_{\varepsilon' B}\| + \|\mathbf{1}_{\varepsilon A}\| + \|\mathbf{1}_{\varepsilon' B}\| \\
&\leq \|x + \mathbf{1}_{\varepsilon' B}\| + (1 + C_{sd})\|\mathbf{1}_{\varepsilon' B}\| \\
&\leq \|x + \mathbf{1}_{\varepsilon' B}\| + (1 + C_{sd})C_{ql}\|x + \mathbf{1}_{\varepsilon' B}\| \\
&= (1 + (1 + C_{sd})C_{ql})\|x + \mathbf{1}_{\varepsilon' B}\|.
\end{aligned}$$

Finally, we present the characterization a) $\Leftrightarrow$ d). Assume a). Since in (2.20) we can take $|A| = |B|$, we can apply the inequality (2.20) for the element $\frac{x}{t}$ and for sets with $|A| = |B|$ and we obtain d). Now, assume d) and take $(A, B, \varepsilon, \varepsilon', x) \in \mathfrak{F}_d$ and select a set C such that $C \cap (A \cup B \cup \operatorname{supp}(x)) = \emptyset$ with $|C \cup A| = |B|$. Applying (2.26) with $t = 1$,

$$\|x + \mathbf{1}_{\varepsilon A}\| \leq \frac{1}{2}\left(\|x + \mathbf{1}_{\varepsilon A} + \mathbf{1}_C\| + \|x + \mathbf{1}_{\varepsilon A} - \mathbf{1}_C\|\right) \leq C_2\|x + \mathbf{1}_{\varepsilon' B}\|.$$

The proof is over. □

Now, we present an example that can be found in [21] of a super-democratic basis that is not symmetric for largest coefficients (and then, it is not quasi-greedy for largest coefficients).

Example 2.2.9. Let $\mathbb{X} = \ell_1 \oplus c_0$ with $\|(x,y)\| = \|x\|_1 + \|y\|_\infty$. Let $(\mathbf{e}_n)_n$ and $(\mathbf{f}_m)_m$ be the canonical bases of ℓ_1 and c_0, respectively. We define

$$E_{2n-1} = \left(\frac{1}{2}\mathbf{e}_n, \frac{-1}{2}\mathbf{f}_n\right),\ E_{2n} = \left(\frac{1}{4}\mathbf{e}_n, \frac{3}{4}\mathbf{f}_n\right),\ n = 1, 2, \ldots,$$

and consider $\mathscr{B} = \{E_n\}_n = \{E_{2n-1}, E_{2n}\}_n$. For this basis we have that $\|E_n\| = 1$ for all $n \in \mathbb{N}$. The biorthogonal dual system $\mathscr{B}^*$ is given by

$$E^*_{2n-1}(x,y) = \frac{3}{2}\mathbf{e}^*_n(x) - \frac{1}{2}\mathbf{f}^*_n(y),\ E^*_{2n}(x,y) = \mathbf{e}^*_n(x) + \mathbf{f}^*_n(y),\ n = 1, 2, \ldots$$

In $\mathbb{X}^* = \ell_\infty \oplus_\infty \ell_1$, $\mathscr{B}^* = \{E^*_n\}_{n=1}^\infty$ defined by

$$E^*_{2n-1} = \left(\frac{3}{2}\mathbf{e}^*_n, -\frac{1}{2}\mathbf{f}^*_n\right),\ E^*_{2n} = \left(\mathbf{e}^*_n, \mathbf{f}^*_n\right),\ n = 1, 2, \ldots,$$

is a basic sequence of $\mathbb{X}^*$. For $\mathscr{B}^*$, we have that $\|E^*_{2n-1}\|_* = 3/2$ and $\|E^*_{2n}\|_* = 1$, $n = 1, 2, \ldots$.

To prove that $\mathscr{B}$ is super-democratic, we show the following proposition:

Proposition 2.2.10. For the basis $\mathscr{B} = \{E_n\}_n$ in $\mathbb{X} = \ell_1 \oplus c_0$ defined above, $\|\mathbf{1}_{\varepsilon A}\| \approx |A|$ for any set $A \in \mathbb{N}^{<\infty}$ and any $\varepsilon \in \Psi_A$.

Proof. By Lemma 1.2.2, $\|\mathbf{1}_{\varepsilon A}\| \leq \frac{3}{2}|A|$. We prove now that $\|\mathbf{1}_{\varepsilon A}\| \geq \frac{1}{8}|A|$. To this end, given $A \subset \mathbb{N}$ finite, we write

$$\begin{aligned} A_1 &= \{k \in \mathbb{N} : 2k \in A \text{ and } 2k-1 \in A\},\\ A_2 &= \{k \in \mathbb{N} : 2k \in A \text{ and } 2k-1 \notin A\},\\ A_3 &= \{k \in \mathbb{N} : 2k \notin A \text{ and } 2k-1 \in A\}. \end{aligned}$$

Observe that the sets A_1, A_2, A_3 are mutually disjoint, and $2|A_1| + |A_2| + |A_3| = |A|$. For any choice of signs,

$$\begin{aligned} \|\mathbf{1}_{\varepsilon A}\| &= \left\| \sum_{k \in A_1} \varepsilon_{2k} E_{2k} + \varepsilon_{2k-1} E_{2k-1} + \sum_{k \in A_2} \varepsilon_{2k} E_{2k} + \sum_{k \in A_3} \varepsilon_{2k-1} E_{2k-1} \right\| \\ &\geq \sum_{k \in A_1} \left| \frac{1}{4}\varepsilon_{2k} + \frac{1}{2}\varepsilon_{2k-1} \right| + \sum_{k \in A_2} \frac{1}{4} + \sum_{k \in A_3} \frac{1}{2}. \end{aligned}$$

Therefore,

$$\|\mathbf{1}_{\varepsilon A}\| \geq \frac{1}{4}|A_1| + \frac{1}{4}|A_2| + \frac{1}{2}|A_3| \geq \frac{1}{8}|A|.$$

This finishes the proof.

$\square$

Back to Example 2.2.9: to verify that $\mathscr{B}$ is not symmetric for largest coefficients, consider the following element: $z = \sum_{n=1}^{N} 2E_{2n} - \sum_{n=1}^{N} E_{2n-1}$. Then,

$$\|z\| = \left\| \sum_{n=1}^{N} (0, 2\mathbf{f}_n) \right\| = 2.$$

Now, consider $z' = \sum_{i=1}^{N} 2E_{2i+4N} - \sum_{i=1}^{N} E_{2i-1}$, then

$$\|z'\| = \left\| \sum_{i=1}^{N} \left(\frac{1}{2}\mathbf{e}_{2i+4N}, \frac{3}{2}\mathbf{f}_{2i+4N} \right) - \sum_{i=1}^{N} \left(\frac{1}{2}\mathbf{e}_i, \frac{-1}{2}\mathbf{f}_i \right) \right\| = \frac{1}{2}(2N+3).$$

Hence, the basis is not symmetric for largest coefficients and then, using Theorem 2.2.8, the basis is not quasi-greedy for largest coefficients.

To close this section, we study the relation between super-democracy and disjoint-super-democracy. It is clear that these notions are equivalent and $C_{sd} \leq C_s \leq C_{sd}^2$. But now, we present a result, proved for the first time in [20], showing that for Schauder bases, $C_{sd} \approx C_s$.

Theorem 2.2.11. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is an SM-Schauder basis in a Banach space $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant defined in (1.7) and $\varkappa = \sup_n \|\mathbf{e}_n\| \|\mathbf{e}_n^*\|_*$. Then, super-democracy and disjoint-super-democracy are equivalent notions with

$$C_{sd} \leq C_s \leq 2(\mathfrak{K}_b + 1)C_{sd} + \varkappa \mathfrak{K}_b. \tag{2.27}$$

Proof. Of course, $C_{sd} \leq C_s$. Now, take $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}'$. Then,

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} \leq \frac{\|\mathbf{1}_{\varepsilon(A \setminus B)}\|}{\|\mathbf{1}_{\varepsilon' B}\|} + \frac{\|\mathbf{1}_{\varepsilon(A \cap B)}\|}{\|\mathbf{1}_{\varepsilon' B}\|} = I + II.$$

Of course, $I \leq C_{sd}$. We only have to give a bound for II. Pick an integer n_0 such that $B_1 = \{n \in B : n \leq n_0\}$ and $B_2 = B \setminus B_1$ satisfy

$$|B_1| = |B_2| \text{ (if } |B| \text{ is even), or } |B_1| = \frac{|B|-1}{2} = |B_2| - 1 \text{ (if } |B| \text{ is odd).}$$

Then,

$$\begin{aligned} II &\leq \frac{\|\mathbf{1}_{\varepsilon(A \cap B_1)}\|}{\|\mathbf{1}_{\varepsilon' B}\|} + \frac{\|\mathbf{1}_{\varepsilon(A \cap B_2)}\|}{\|\mathbf{1}_{\varepsilon' B}\|} \\ &\leq (\mathfrak{K}_b + 1)\frac{\|\mathbf{1}_{\varepsilon(A \cap B_1)}\|}{\|\mathbf{1}_{\varepsilon' B_2}\|} + \mathfrak{K}_b \frac{\|\mathbf{1}_{\varepsilon(A \cap B_2)}\|}{\|\mathbf{1}_{\varepsilon' B_1}\|} = II_1 + II_2. \end{aligned}$$

Since $|A\cap B_1|\le |B_1|\le |B_2|$, we see that

$$II_1 \le (\mathfrak{K}_b+1)C_{sd}.$$

On the other hand, picking any number $n_1\in A\cap B_2$ and using that $1=|\mathbf{e}^*_{n_1}(\mathbf{1}_{\varepsilon' B})|\le \|\mathbf{1}_{\varepsilon' B}\|\|\mathbf{e}^*_{n_1}\|_*$,

$$II_2 \le \mathfrak{K}_b \frac{\|\mathbf{1}_{\varepsilon(A\cap B_2\setminus\{n_1\})}\|}{\|\mathbf{1}_{\varepsilon' B_1}\|}+\mathfrak{K}_b\|\mathbf{e}_{n_1}\|\|\mathbf{e}^*_{n_1}\|_*\le \mathfrak{K}_b C_{sd}+\varkappa\mathfrak{K}_b,$$

where the last bound is true since $|A\cap B_2\setminus\{n_1\}|\le |B_2|-1\le |B_1|$. Hence, due to the bounds of I, II, II_1 and II_2, we obtain the result.

□

2.3 Greedy bases

From a theoretical point of view, it is interesting to know if a particular algorithm of approximation $(T_m)_m$ produces the best possible approximation, i.e., when $\|x-T_m(x)\|$ is comparable to the *m-th best possible error of approximation* $\sigma_m(x)$ for all $x\in\mathbb{X}$ and $m\in\mathbb{N}$. In the case of the TGA, this idea invokes the notion of greediness. If $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$, for each m, we define

$$\Sigma_m[\mathscr{B},\mathbb{X}] := \Sigma_m = \left\{\sum_{n\in A} a_n\mathbf{e}_n : A\subset\mathbb{N}, |A|\le m, a_n\in\mathbb{F}\right\},$$

and

$$\sigma_m(x,\mathscr{B})_{\mathbb{X}} := \sigma_m(x) = \operatorname{dist}(x,\Sigma_m) = \inf\left\{\left\|x-\sum_{n\in A} a_n\mathbf{e}_n\right\| : A\subset\mathbb{N}, |A|\le m, a_n\in\mathbb{F}\right\}.$$

Definition 2.3.1 ([60]). An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *greedy* if there exists a positive constant C such that

$$\|x-\mathscr{G}_m(x)\|\le C\sigma_m(x),\ \forall x\in\mathbb{X}, \forall m\in\mathbb{N}. \tag{2.28}$$

The least constant C in (2.28) is denoted by $C_g[\mathscr{B},\mathbb{X}] := C_g$ and we will say that $\mathscr{B}$ is *C_g-greedy*.

Example 2.3.2. Some examples of greedy bases are the following:

- The canonical basis $\mathscr{B}=(\mathbf{e}_n)_n$ of the space ℓ_p, $1\le p<\infty$ is 1-greedy: let $x\in\ell_p$, $m\in\mathbb{N}$ and $G=\{\rho(1),...,\rho(m)\}$ with ρ the natural greedy ordering of x. Let $z=(z_j)_j\in\ell_p$ such that $\operatorname{supp}(z)=B$, $|B|\le m$ and $\|x-z\|<\sigma_m(x)+\varepsilon$ with $\varepsilon>0$. Then,

$$\begin{aligned}\|x-\mathscr{G}_m(x)\|^p &= \sum_{n\in\mathbb{N}\setminus G}|\mathbf{e}^*_n(x)|^p \le \sum_{n\in\mathbb{N}\setminus G}|\mathbf{e}^*_n(x)|^p+\sum_{n\in B}|\mathbf{e}^*_n(x)-z_j|^p\\ &\le \sum_{n\in\mathbb{N}\setminus B}|\mathbf{e}^*_n(x)|^p+\sum_{n\in B}|\mathbf{e}^*_n(x)-z_j|^p = \|x-z\|^p<(\sigma_m(x)+\varepsilon)^p.\end{aligned}$$

Hence, the basis is 1-greedy.

- The Haar basis in $L_p([0,1))$ is greedy for $1 < p < \infty$. This is a celebrated theorem of V. N. Temlyakov (see [77]). We will discuss in Chapter 3 how the greedy constant C_g associated to the Haar system behaves.

S. V. Konyagin and V. N. Temlyakov were the first authors that gave a characterization of greedy bases in terms of unconditional and democratic bases.

Theorem 2.3.3 ([60, Theorem 1])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is greedy if and only if $\mathscr{B}$ is democratic and unconditional. Also,

$$\max\{K_{su}, C_d\} \leq C_g \leq K_{su} + K_{su}K_u^2 C_d.$$

In the last theorem we see that if $C_d = K_u = 1$, the greedy constant is $C_g \leq 2$ and, also, this bound is sharp since F. Albiac and P. Wojtaszczyk proved in [11] that there exists a basis in a Banach space such that $K_u = C_d = 1$ and $C_g = 2$. With the objective to recover $C_g = 1$, they introduced the so called Property (A).

An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ has the Property (A) if for any $x \in \mathbb{X}$,

$$\|x\| = \left\| \sum_{n \in M(x)} \theta_n \mathbf{e}_n^*(x) \mathbf{e}_{\pi(n)} + (x - P_{M(x)}(x)) \right\|, \tag{2.29}$$

for $M(x) = \{n \in \operatorname{supp}(x) : |\mathbf{e}_n^*(x)| = \max_n |\mathbf{e}_n^*(x)|\}$ and for all injective maps $\pi : \operatorname{supp}(x) \to \mathbb{N}$ such that $\pi(j) = j$ if $j \notin M(x)$, $\theta \in \Psi_{M(x)}$ with $\theta_n = 1$ whenever $\pi(n) = n$ for $n \in M(x)$.

Remark 2.3.4. F. Albiac and P. Wojtaszczyk introduced the Property (A) for the real case.

Theorem 2.3.5 ([11, Theorem 3.4])**.** An SM-basis $\mathscr{B}$ in a real Banach space $\mathbb{X}$ is 1-greedy if and only if $\mathscr{B}$ is 1-unconditional and has the Property (A).

Later on, S. J. Dilworth, D. Kutzarova, E. Odell, T. Schlumprecht and A. Zsák ([34]) generalized the notion of Property (A) to give a general characterization of greedy bases with the possibility to recover the constant one. The notion that they introduced was the symmetry for largest coefficients, that is, the condition (2.20). We present a proof that can be found in [16], showing that Property (A) and 1-symmetric for largest coefficients are equivalent notions.

Lemma 2.3.6. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in Banach space. $\mathscr{B}$ has the Property (A) if and only if $\mathscr{B}$ is 1-symmetric for largest coefficients.

Proof. Assume that $\mathscr{B}$ has the Property (A). For each $(A, B, \varepsilon, \varepsilon', x) \in \mathfrak{F}_d$ with $|A| = |B|$, we write $y = \mathbf{1}_{\varepsilon A} + x$. Hence $M(y) = A \cup \{n \in \operatorname{supp}(x) : |\mathbf{e}_n^*(x)| = 1\}$. Let $\pi : A \to B$ be a bijection and set $\theta_n(y) = \varepsilon'_{\pi(n)}$ for $n \in A$. Using Property (A), $\|y\| = \|\mathbf{1}_{\varepsilon' B} + x\|$.

Conversely given $x \in \mathbb{X}$ and $\alpha = \max_n |\mathbf{e}_n^*(x)|$ one can consider, for each π and θ in the conditions of the Property (A), the set $A = \{j \in M(x) : \pi(j) \neq j\}$ and define $\varepsilon_n = \frac{\mathbf{e}_n^*(x)}{|\mathbf{e}_n^*(x)|}$ for each $n \in A$. Now, selecting $B = \pi(A)$ and $\varepsilon'_n = \theta_n$ for $n \in B$, we have

$$\begin{aligned} \|x\| &= \alpha \left\| \mathbf{1}_{\varepsilon A} + \frac{1}{\alpha}(x - P_A(x)) \right\| \\ &= \alpha \left\| \mathbf{1}_{\varepsilon' B} + \frac{1}{\alpha}(x - P_A(x)) \right\| \\ &= \left\| \sum_{n \in A} \theta_n \mathbf{e}_n^*(x) e_{\pi(n)} + (x - P_A(x)) \right\|. \end{aligned}$$

□

The purpose of this section is to establish the characterization of greedy bases in terms of unconditionality, super-democracy and symmetry for largest coefficients giving a slightly shorter proof for Theorem 2.3.3 and, also, improving the behavior of the constants with the possibility to recover the constant $C_g = 1$. Our proof uses some standard techniques (see [4, 18, 34, 60]).

Theorem 2.3.7. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_g-greedy, then the basis is K_{su}-unconditional and C_a-SLC with

$$\max\{K_{su}, C_a\} \leq C_g.$$

b) If $\mathscr{B}$ is C_{sd}-disjoint-super-democratic and K_{su}-unconditional, then $\mathscr{B}$ is C_g-greedy with

$$C_g \leq K_{su}(1 + C_{sd}).$$

c) If $\mathscr{B}$ is K_{su}-unconditional and C_a-SLC, then the basis is C_g-greedy with

$$C_g \leq K_{su} C_a.$$

In particular, if $K_{su} = C_a = 1$, the basis is 1-greedy.

Proof. a) Assume that $\mathscr{B}$ is C_g-greedy. Let $x \in \mathbb{X}$ with $\operatorname{supp}(x) = B$, $|B| < \infty$ and $A \subset B$. Define now $y := P_A(x) + \sum_{n \in B \setminus A} (\alpha + \mathbf{e}_n^*(x)) \mathbf{e}_n$, where

$$\alpha > \sup_{n \in A} |\mathbf{e}_n^*(x)| + \sup_{n \in B \setminus A} |\mathbf{e}_n^*(x)|.$$

A is the k-th greedy set for y for $k := |B \setminus A|$, then

$$\|P_A(x)\| = \|y - \mathscr{G}_k(y)\| \leq C_g \sigma_k(y) \leq C_g \|y - \alpha \mathbf{1}_{B \setminus A}\| = C_g \|x\|.$$

Thus, by density, the basis is unconditional with $K_{su} \leq C_g$.

To show now the SLC, take $(A, B, \varepsilon, \varepsilon', x) \in \mathfrak{F}_d$. Set $y := x + \mathbf{1}_{\varepsilon A} + (1+\delta)\mathbf{1}_{\eta B}$ with $\delta > 0$. If $k := |B|$,

$$\|x + \mathbf{1}_{\varepsilon A}\| = \|y - \mathscr{G}_k(y)\| \leq C_g \sigma_k(y) \leq C_g \|y - \mathbf{1}_{\varepsilon A}\| = C_g \|x + (1+\delta)\mathbf{1}_{\varepsilon' B}\|.$$

Taking $\delta \searrow 0$, we obtain that the basis is SLC with $C_a \le C_g$.

b) Assume that $\mathscr{B}$ is K_{su}-unconditional and C_{sd}-disjoint-super-democratic. Take $x \in \mathbb{X}$ and $\mathscr{G}_m(x)$ the m-th greedy sum of x with $A = \text{supp}(\mathscr{G}_m(x))$. For each $\eta > 0$ find $y \in \mathbb{X}$ such that $\|x-y\| < \sigma_m(x) + \eta$, with $\text{supp}(y) = B$ and $|B| \le m$. We have the following decomposition:

$$x - \mathscr{G}_m(x) = P_{(A\cup B)^c}(x-y) + P_{B\setminus A}(x).$$

On the one hand, by Remark 1.1.4, $\|P_{(A\cup B)^c}(x-y)\| \le K_{su}\|x-y\|$, so we only have to estimate $\|P_{B\setminus A}(x)\|$. First of all, by (i) of Corollary 1.3.3,

$$\|P_{B\setminus A}(x)\| \le \max_{j\in B\setminus A} |\mathbf{e}_j^*(x)| \sup_{\varepsilon'\in\Psi_{B\setminus A}} \|\mathbf{1}_{\varepsilon'(B\setminus A)}\|. \tag{2.30}$$

Take $\varepsilon \equiv \text{sign}\{(\mathbf{e}_j^*(x-y))\}$. Since $|B\setminus A| \le |A\setminus B|$ and the basis is disjoint-super-democratic,

$$\sup_{\varepsilon'\in\Psi_{B\setminus A}} \|\mathbf{1}_{\varepsilon'(B\setminus A)}\| \le C_{sd}\|\mathbf{1}_{\varepsilon(A\setminus B)}\|.$$

Hence, using this in (2.30),

$$\begin{aligned}\|P_{B\setminus A}(x)\| &\le C_{sd} \max_{j\in B\setminus A} |\mathbf{e}_j^*(x)| \|\mathbf{1}_{\varepsilon(A\setminus B)}\| \\ &\le C_{sd} \min_{j\in A\setminus B} |\mathbf{e}_j^*(x)| \|\mathbf{1}_{\varepsilon(A\setminus B)}\|. \end{aligned}\tag{2.31}$$

Now, we write

$$\min_{j\in A\setminus B} |\mathbf{e}_j^*(x)| \mathbf{1}_{\varepsilon(A\setminus B)} = \sum_{i\in A\setminus B} \lambda_i \mathbf{e}_i^*(x-y)\mathbf{e}_i,$$

where $\lambda_i = \dfrac{\min_{j\in(A\setminus B)} |\mathbf{e}_j^*(x-y)|}{|\mathbf{e}_i^*(x-y)|}$ and $0 < \lambda_i \le 1$. Using now Theorem 1.3.4 and the unconditionality,

$$\left\|\sum_{i\in A\setminus B} \lambda_i \mathbf{e}_i^*(x-y)\mathbf{e}_i\right\| \le \sup_{D\subseteq A\setminus B} \left\|\sum_{i\in D} \mathbf{e}_i^*(x-y)\mathbf{e}_i\right\| \le K_{su}\|x-y\|. \tag{2.32}$$

Hence, using (2.31) and (2.32), we obtain that $\mathscr{B}$ is greedy with $C_g \le K_{su}(1+C_{sd})$.

c) Finally, we prove that if $\mathscr{B}$ is K_{su}-unconditional and C_a-SLC, then it is greedy. Take x, $\mathscr{G}_m(x)$ and y as in b) and define $t := \min\{|\mathbf{e}_n^*(x)| : n \in A\}$ and $\eta \equiv \{\text{sign}(\mathbf{e}_n^*(x))\}$. Using the item b) of Proposition 2.2.8,

$$\begin{aligned}\|x - \mathscr{G}_m(x)\| &\le C_a\|x - P_A(x) - P_{B\setminus A}(x) + t\mathbf{1}_{\eta(A\setminus B)}\| \\ &= C_a\|P_{(A\cup B)^c}(x-y) + t\mathbf{1}_{\eta(A\setminus B)}\|. \end{aligned}\tag{2.33}$$

Now, taking into account that $P_{(A\cup B)^c}(x-y) + t\mathbf{1}_{\eta(A\setminus B)} = T_t((x-y) - P_B(x-y))$, by (2.9),

$$\|T_t((x-y) - P_B(x-y))\| \le K_{su}\|x-y\|. \tag{2.34}$$

By (2.33) and (2.34) the proof is over. □

2.4 The best m-th error of approximation using polynomials with constant coefficients

In this section we study an equivalence of greediness from a new point of view as we have done in [16, 17]. For this, define the *best m-th error of approximation with polynomials of constant coefficients* $\mathscr{D}_m(x)$, by

$$\mathscr{D}_m[\mathscr{B},\mathbb{X}](x) := \mathscr{D}_m(x) = \inf\{\|x-\alpha\mathbf{1}_{\varepsilon A}\| : A\subset\mathbb{N}, |A|=m, \alpha\in\mathbb{F}, \varepsilon\in\Psi_A\}.$$

Definition 2.4.1. An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *greedy for polynomials with constant coefficients* (GPCC for short) if there is a positive constant C such that

$$\|x-\mathscr{G}_m(x)\| \leq C\mathscr{D}_m(x),\ \forall x\in\mathbb{X}, \forall m\in\mathbb{N}, \tag{2.35}$$

The least constant C in (2.35) is denoted by $C_{pg}[\mathscr{B},\mathbb{X}] := C_{pg}$ and we will say that $\mathscr{B}$ has the *C_{pg}-GPCC property*.

The main result that we show is the following and can be found in [17].

Theorem 2.4.2. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$.

- If $\mathscr{B}$ is C_g-greedy, then $\mathscr{B}$ has the C_{pg}-GPCC property with $C_{pg}\leq C_g$.
- If $\mathscr{B}$ has the C_{pg}-GPCC property, then $\mathscr{B}$ is C_g-greedy with $C_g\leq (C_{pg})^2$.

Proof. Since $\sigma_m(x)\leq\mathscr{D}_m(x)$ for all $x\in\mathbb{X}$ and $m\in\mathbb{N}$, we only have to show that the GPCC property implies greediness. Let $x\in\mathbb{X}$, $m\in\mathbb{N}$ and $\mathscr{G}_m(x)=P_A(x)$. For each $\varepsilon>0$ we choose $y\in\mathbb{X}$ verifying $y=\sum_{n\in B} b_n\mathbf{e}_n$ with $|B|\leq m$ and $\|x-y\|<\sigma_m(x)+\varepsilon$. We can assume without loss of generality that $\sup_n|\mathbf{e}_n^*(x)|=1$. We can write

$$x-P_A(x) = x-P_{A\cup B}(x)+P_{B\setminus A}(x).$$

If we put $\gamma=\max\limits_{j\in B\setminus A}|\mathbf{e}_j^*(x)|$, taking into account that

$$P_{B\setminus A}(x) = \gamma\sum_{j\in B\setminus A}\frac{|\mathbf{e}_j^*(x)|}{\gamma}\eta_j\mathbf{e}_j,$$

where $\eta\equiv\{\mathrm{sign}(\mathbf{e}_j^*(x))\}$, by Corollary 1.3.2,

$$\|x-P_{A\cup B}(x)+P_{B\setminus A}(x)\| \leq \sup_{\varepsilon'\in\Psi_{B\setminus A}}\|x-P_{A\cup B}(x)+\gamma\sum_{n\in B\setminus A}\varepsilon_j'\mathbf{e}_j\|. \tag{2.36}$$

Then, we only have to show that, for any $\varepsilon'\in\Psi_{B\setminus A}$,

$$\|x-P_{A\cup B}(x)+\gamma\mathbf{1}_{\varepsilon'(B\setminus A)}\| \leq C_{pg}^2\|x-y\|. \tag{2.37}$$

For $\delta > 0$, consider the element

$$z_\eta = \sum_{n\in A\setminus B}(\mathbf{e}_n^*(x)+\delta\eta_n)\mathbf{e}_n + \sum_{n\in(A\cup B)^c}\mathbf{e}_n^*(x)\mathbf{e}_n + \sum_{n\in B\setminus A}\gamma\varepsilon'_n\mathbf{e}_n.$$

Note that

$$\min_{n\in A\setminus B}|\mathbf{e}_n^*(z_\eta)| > \min_{n\in A\setminus B}|\mathbf{e}_n^*(x)| \geq \min_{n\in A}|\mathbf{e}_n^*(x)|$$

and

$$\max_{n\in(A\setminus B)^c}|\mathbf{e}_n^*(z_\eta)| \leq \max_{n\notin A}|\mathbf{e}_n^*(x)|.$$

Therefore, we conclude that

$$\min_{n\in A\setminus B}|\mathbf{e}_n^*(z_\eta)| > \max_{n\in(A\setminus B)^c}|\mathbf{e}_n^*(z_\eta)|.$$

Hence $A\setminus B$ is k-th greedy set of z_η for $k := |A\setminus B| = |B\setminus A|$ since $|B| = m$. Then, notice that

$$z_\eta - \mathscr{G}_k(z_\eta) = x - P_{A\cup B}(x) + \gamma\mathbf{1}_{\varepsilon'(B\setminus A)}.$$

Thus,

$$\begin{aligned}
\|x-P_{(A\cup B)}(x)+\gamma\mathbf{1}_{\varepsilon'(B\setminus A)}\| &\leq C_{pg}\mathscr{D}_k(z_\eta)\\
&\leq C_{pg}\|z_\eta-\gamma\mathbf{1}_{\varepsilon'(B\setminus A)}\|\\
&= C_{pg}\|\sum_{n\in A\setminus B}(\mathbf{e}_n^*(x)+\delta\eta_n)\mathbf{e}_n+\sum_{n\in(A\cup B)^c}\mathbf{e}_n^*(x)\mathbf{e}_n\|.
\end{aligned}$$

Taking limits when δ goes to 0,

$$\|x-P_{(A\cup B)}(x)+\gamma\mathbf{1}_{\varepsilon'(B\setminus A)}\| \leq C_{pg}\|x-P_B(x)\| \tag{2.38}$$

Now, let $z = x-y+\mu\mathbf{1}_B$ for $\mu > \max_{j\notin B}|\mathbf{e}_j^*(x-y)| + \max_{j\in B}|\mathbf{e}_j^*(x-y)|$. It is clear that B is the m-th greedy set of z. Hence

$$\|x-P_B(x)\| = \|z-\mathscr{G}_n(z)\| \leq C_{pg}\|z-\mu\mathbf{1}_B\| = C_{pg}\|x-y\|. \tag{2.39}$$

Therefore, by (2.38) and (2.39) we obtain

$$\|x-P_{(A\cup B)}(x)+\gamma\mathbf{1}_{\varepsilon'(B\setminus A)}\| \leq C_{pg}^2\|x-y\|.$$

□

Corollary 2.4.3. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$. The following are equivalent:

a) $\mathscr{B}$ is greedy.

b) There exists a positive constant C such that for all $x\in\mathbb{X}$ and $m\in\mathbb{N}$,

$$\|x-\mathscr{G}_m(x)\| \leq C\inf\{\|x-\alpha\mathbf{1}_{\varepsilon A}\| : \alpha\in\mathbb{F}, A\subset\mathbb{N}, |A|\leq m, \varepsilon\in\Psi_A\}.$$

c) $\mathscr{B}$ has the GPCC-property.

d) $\mathscr{B}$ is unconditional and symmetric for largest coefficients.

Proof. The implications a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are trivial by the definition of $\sigma_m(x)$ and $\mathscr{D}_m(x)$. Now, c) $\Rightarrow$ a) is a consequence of Theorem 2.4.2. To show that c) $\Rightarrow$ d), we only have to apply the same proof than in the item a) of Theorem 2.3.7 and d) $\Rightarrow$ a) is the item c) of Theorem 2.3.7. □

2.4.1 Properties of the error $\mathscr{D}_m(x)$

Theorem 2.4.2 is somehow "strange" because while $\sigma_m(x) \to 0$ in any Banach space, $\mathscr{D}_m(x)$ has not this behavior. To show this fact, first, we prove that $\liminf_{m\to+\infty} \mathscr{D}_m(x)$ can not be zero and, also, we give a characterization showing when there exists a positive constant c such that

$$c\|x\| \leq \liminf_{m\to+\infty} \mathscr{D}_m(x) \leq \limsup_{m\to+\infty} \mathscr{D}_m(x) \leq \|x\|,\ \forall x \in \mathbb{X}. \tag{2.40}$$

To show the existence of the constant c in (2.40), we will use the functions $\varphi_u(m)$ and $h_l(m)$. These results can be found in [22].

Proposition 2.4.4. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-Schauder basis of a Banach space $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant. Then, for every $x \in \mathbb{X}$,

$$\frac{1}{4\mathfrak{K}_b} \sup_{n\in\mathbb{N}} |\mathbf{e}_n^*(x)| \leq \liminf_{m\to\infty} \mathscr{D}_m(x).$$

Proof. Let $x \in \mathbb{X}$. Note that for every finite set $A \subset \mathbb{N}$, $\alpha \in \mathbb{F}$ and $\varepsilon \in \Psi_A$ it holds that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \sup_{n\in\mathbb{N}} \frac{|\mathbf{e}_n^*(x-\alpha\mathbf{1}_{\varepsilon A})|}{\|\mathbf{e}_n^*\|_*} \geq \frac{\sup_{n\in\mathbb{N}} |\mathbf{e}_n^*(x-\alpha\mathbf{1}_{\varepsilon A})|}{2\mathfrak{K}_b} \geq \frac{\sup_{n\in\mathbb{N}} \big||\mathbf{e}_n^*(x)| - |\alpha|\big|}{2\mathfrak{K}_b}.$$

Let us also fix $\delta > 0$ and $n_0 \in \mathbb{N}$ with the property that

$$|\mathbf{e}_n^*(x)| \leq \delta \quad \text{for every } n \geq n_0.$$

If A satisfies $|A| > n_0$, then there is $j \in A$ with $j > n_0$, and so

$$\|x - \alpha\mathbf{1}_{\varepsilon A}\| \geq \frac{|\mathbf{e}_j^*(x) - |\alpha||}{2\mathfrak{K}_b} \geq \frac{||\alpha| - \delta|}{2\mathfrak{K}_b}.$$

In particular, combining both lower estimations we get that for $|A| > n_0$

$$\|x - \alpha\mathbf{1}_{\varepsilon A}\| \geq \frac{||\alpha|-\delta| + \sup_{n\in\mathbb{N}} \big||\mathbf{e}_n^*(x)| - |\alpha|\big|}{4\mathfrak{K}_b} \geq \sup_{n\in\mathbb{N}} \frac{|\mathbf{e}_n^*(x)| - \delta}{4\mathfrak{K}_b}.$$

Therefore, for $m > n_0$

$$\mathscr{D}_m(x) \geq \sup_{n\in\mathbb{N}} \frac{|\mathbf{e}_n^*(x)| - \delta}{4\mathfrak{K}_b}.$$

□

Definition 2.4.5. The functions $h_l(m)$ and $\varphi_u(m)$ are said to be *admissible* if they are both bounded or divergent to infinity.

The main result of the section is the following theorem.

Theorem 2.4.6. Let $\mathscr{B}$ be an SM-Schauder basis of a Banach space $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant. The following assertions are equivalent:

(i) There is a positive constant $c > 0$ such that

$$c\,\|x\| \leq \liminf_{m\to+\infty} \mathscr{D}_m(x) \leq \limsup_{m\to+\infty} \mathscr{D}_m(x) \leq \|x\| \quad \text{for every } x \in \mathbb{X}.$$

(ii) $h_l(m)$ and $\varphi_u(m)$ are admissible.

Moreover, if $\mathscr{B}$ is monotone, that is, $\mathfrak{K}_b = 1$, and $h_l(m) \to +\infty$ as $m \to +\infty$, then

$$\lim_{m\to+\infty} \mathscr{D}_m(x) = \|x\|. \tag{2.41}$$

We divide the proof of this theorem in two propositions.

Proposition 2.4.7. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-Schauder basis of a Banach space $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant. Then,

$$\sup_{\substack{A\subset\mathbb{N} \\ \mathit{finite}, \eta\in\Psi_A}} \liminf_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_{\eta A}) \;\leq\; (1+\mathfrak{K}_b) \liminf_{m\to+\infty} h_l(m) \;\leq\; \infty. \tag{2.42}$$

Proof. Let us fix a finite set $A \subset \mathbb{N}$ and $\eta \in \Psi_A$. By Proposition 2.4.4, we can find $\lambda \in \mathbb{F}$ satisfying

$$\lambda < \liminf_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_{\eta A}). \tag{2.43}$$

We can then find $m_0, n_0 \in \mathbb{N}$ with the following properties:

- $\lambda \leq \|\mathbf{1}_{\eta A} - \alpha \mathbf{1}_{\varepsilon B}\|$ for every $\alpha \in \mathbb{F}$, $\varepsilon \in \Psi_B$ and $B \subset \mathbb{N}$ with $|B| \geq m_0$,
- $A \subset \{1, \ldots, n_0\}$.

Let $C \subset \mathbb{N}$ be a finite set with $|C| \geq m_0 + n_0$. Then,

$$\mathbf{1}_{\varepsilon C} - P_{n_0}(\mathbf{1}_{\varepsilon C}) = \mathbf{1}_{\varepsilon C'}$$

where $C' := C \setminus \{1, \ldots, n_0\}$. Notice that $|C'| \geq m_0$, so in particular

$$\lambda \leq \|\mathbf{1}_{\eta A} - \mathbf{1}_{(\eta A)\cup(\varepsilon C')}\| = \|\mathbf{1}_{\varepsilon C'}\| \leq \|\mathrm{Id}_{\mathbb{X}} - P_{n_0}\|\,\|\mathbf{1}_{\varepsilon C}\| \leq (1+\mathfrak{K}_b)\,\|\mathbf{1}_{\varepsilon C}\|.$$

Thus, we have the relation

$$\lambda \leq (1+\mathfrak{K}_b) \liminf_{m\to+\infty} h_l(m).$$

Taking the supremum on λ according to (2.43) we conclude that

$$\liminf_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_{\eta A}) \;\leq\; (1+\mathfrak{K}_b) \liminf_{m\to+\infty} h_l(m).$$

□

Proposition 2.4.8. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-Schauder basis of a Banach space $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant. Assume that there is a constant $C > 0$ satisfying

$$\sup_{n\in\mathbb{N}} \varphi_u(n) \le C \sup_{n\in\mathbb{N}} h_l(n) \le \infty.$$

Then, for every $x \in \mathbb{X}$

$$\frac{1}{C+\mathfrak{K}_b(1+C)} \|x\| \le \liminf_m \mathscr{D}_m(x) \le \limsup_m \mathscr{D}_m(x) \le \|x\|. \tag{2.44}$$

Proof. Let us fix $x \in \mathbb{X}$. We just have to show that the left hand-side of (2.44) holds. Let $0 < \delta < 1$ and $m_0, n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|P_n(x) - x\| &\le \delta \|x\| \quad \text{for every } n \ge n_0, \\ \varphi_u(n_0) &\le C(1-\delta) h_l(m_0). \end{aligned}$$

Given $\alpha \in \mathbb{F}$, $A \subset \mathbb{N}$ with $|A| \ge m_0 + n_0$ and $\varepsilon \in \Psi_A$, we are going to establish two lower bounds for $\|x - \alpha \mathbf{1}_{\varepsilon A}\|$.

- Since $|A \cap (n_0, +\infty)| \ge m_0$, we can find $n \ge n_0$ such that $|A \cap (n, +\infty)| = m_0$. Thus, applying the operator $\mathrm{Id}_{\mathbb{X}} - P_n$ to $x - \alpha \mathbf{1}_{\varepsilon A}$ we have that

$$\begin{aligned} \|x - \alpha \mathbf{1}_{\varepsilon A}\| &\ge \frac{1}{\mathfrak{K}_b + 1} \|(\mathrm{Id}_{\mathbb{X}} - P_n)(x) - \alpha \mathbf{1}_{\varepsilon(A\cap(n,+\infty))}\| \\ &\ge \frac{1}{\mathfrak{K}_b + 1} \big(|\alpha| h_l(m_0) - \delta \|x\|\big). \end{aligned} \tag{2.45}$$

- As $|A| \ge n_0$ we can find $n \ge n_0$ with $|A \cap [1,n]| = n_0$, so that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \ge \frac{1}{\mathfrak{K}_b} \big(\|P_n(x) - \alpha \mathbf{1}_{\varepsilon(A\cap[1,n])}\|\big) \tag{2.46}$$

$$\ge \frac{1}{\mathfrak{K}_b} \big(\|x\|(1-\delta) - |\alpha| \varphi_u(n_0)\big) \tag{2.47}$$

$$\ge \frac{1-\delta}{\mathfrak{K}_b} \big(\|x\| - C|\alpha| h_l(m_0).\big) \tag{2.48}$$

Note that the lower estimations (2.45) and (2.48) are respectively increasing and decreasing linear functions $f(t)$ and $g(t)$ on $t = |\alpha|$. Moreover these functions have a unique point of intersection $t_0 > 0$ which can be easily checked to satisfy

$$t_0 = \frac{\|x\|}{h_l(m_0)} \cdot \frac{(1-\delta)(1+\mathfrak{K}_b) + \delta \mathfrak{K}_b}{C(1-\delta)(1+\mathfrak{K}_b) + \mathfrak{K}_b}. \tag{2.49}$$

Thus

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \ge \max\{f(|\alpha|), g(|\alpha|)\} \ge f(t_0) = g(t_0) = \frac{\|x\|}{1+\mathfrak{K}_b} \left[\frac{(1-\delta)(1+\mathfrak{K}_b) + \delta \mathfrak{K}_b}{C(1-\delta)(1+\mathfrak{K}_b) + \mathfrak{K}_b} - \delta\right].$$

Taking the infimum of $\|x-\alpha \mathbf{1}_{\varepsilon A}\|$ over $\alpha\in\mathbb{F}$ and A satisfying the conditions above, we deduce that

$$\liminf_{k\to+\infty}\mathscr{D}_k(x)\geq \inf_{k\geq m_0+n_0}\mathscr{D}_k(x)\geq \frac{\|x\|}{1+\mathfrak{K}_b}\left[\frac{(1-\delta)\,(1+\mathfrak{K}_b)+\delta\,\mathfrak{K}_b}{C(1-\delta)(1+\mathfrak{K}_b)+\mathfrak{K}_b}-\delta\right].$$

Finally, making $\delta\to 0^+$ we get the desired conclusion. □

Proof of Theorem 2.4.6. To check (i) ⇒ (ii), note that using Proposition 2.4.7 we then deduce that

$$\sup_{m\in\mathbb{N}}\varphi_u(m)=\sup_{\substack{A\subset\mathbb{N}\\ finite,\eta\in\Psi_A}}\|\mathbf{1}_{\eta A}\|\leq\frac{1}{c}\sup_{\substack{A\subset\mathbb{N}\\ finite,|\eta|=1}}\liminf_{m\to+\infty}\mathscr{D}_m(\mathbf{1}_{\eta A})\leq\frac{(1+\mathfrak{K}_b)}{c}\liminf_{m\to+\infty}h_l(m)\leq\infty.$$

It is clear from this inequality that $h_l(m)$ and $\varphi_u(m)$ are then admissible. To see the converse (ii) ⇒ (i), note first that if $h_l(m)$ and $\varphi_u(m)$ are admissible, then there exists $C>0$ such that

$$\sup_{m\in\mathbb{N}}\varphi_u(m)\leq\sup_{m\in\mathbb{N}}C\,h_l(m) \tag{2.50}$$

and we can apply Proposition 2.4.8 to prove the result. The second statement of the theorem follows also from Proposition 2.4.8 since $\mathscr{B}$ being monotone means that $\mathfrak{K}_b=1$, and condition $\lim_m h_l(m)=+\infty$ means that (2.50) holds for every $C>0$. □

2.5 Almost-greedy bases

In greedy bases we compare the m-th greedy error with $\sigma_m(x)$ for all $x\in\mathbb{X}$ and $m\in\mathbb{N}$, but this comparison is a little bit unfair since for each $x\in\mathbb{X}$ and $m\in\mathbb{N}$, $\mathscr{G}_m(x)$ gives an approximation of the form $P_A(x)$, and $\sigma_m(x)$ contains a bigger class of approximants. For this reason, S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov introduced the notion of *almost-greedy bases*. The definition that they introduced in [32] was the following: an SM-basis is almost-greedy if there exists a positive constant C such that

$$\|x-\mathscr{G}_m(x)\|\leq C\inf\{\|x-P_A(x)\|:A\subset\mathbb{N},|A|=m\}.$$

Here, we consider an alternative definition. For that, we define the error $\widetilde{\sigma}_m(x)$ as

$$\widetilde{\sigma}_m(x,\mathscr{B})_{\mathbb{X}}:=\widetilde{\sigma}_m(x)=\inf\{\|x-P_A(x)\|:A\subset\mathbb{N},|A|\leq m\}.$$

Definition 2.5.1. An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *almost-greedy* if there exists a constant $C\geq 1$ such that

$$\|x-\mathscr{G}_m(x)\|\leq C\widetilde{\sigma}_m(x),\ \forall x\in\mathbb{X},\forall m\in\mathbb{N}. \tag{2.51}$$

The least constant C in (2.51) is denoted by $C_{al}[\mathscr{B},\mathbb{X}]:=C_{al}$ and we will say that $\mathscr{B}$ is C_{al}*-almost-greedy*.

Remark 2.5.2. F. Albiac and J. L. Ansorena proved in [4] that the definition introduced by S. J. Dilworth et al. in [32] is equivalent to the Definition 2.5.1.

The first authors that proved a characterization of almost-greedy bases were S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov in [32] and they established the following theorem for the real case.

Theorem 2.5.3 ([32])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is almost-greedy if and only if $\mathscr{B}$ is democratic and quasi-greedy. Quantitatively:

- If $\mathscr{B}$ is C_{al}-almost-greedy, then $\mathscr{B}$ is C_d-democratic and C_q-quasi-greedy with
$$\max\{C_d, C_q\} \le C_{al}.$$
- If $\mathscr{B}$ is C_d-democratic and C_q-quasi-greedy, $\mathscr{B}$ is C_{al}-almost-greedy with
$$C_{al} \le 8C_q^4 C_d + C_q + 1.$$

As for greedy bases, we present now the characterization of almost-greediness using the symmetry for largest coefficients instead of democracy, since in the above theorem, we can not recover the constant $C_{al} = 1$ using $C_d = C_q = 1$. In fact, the example presented by F. Albiac and P. Wojtaszczyk in [11] shows that $C_q = C_d = 1$ and $C_{al} = 2$.

Theorem 2.5.4. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis of a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_{al}-almost-greedy, then the basis is C_q-quasi-greedy and C_a-symmetric for largest coefficients with
$$\max\{C_q, C_a\} \le C_{al}.$$
b) If $\mathscr{B}$ is C_q-quasi-greedy and C_{sd}-disjoint-super-democratic, then the basis is C_{al}-almost-greedy with
$$C_{al} \le C_q(1 + 2C_{sd}).$$
c) If $\mathscr{B}$ is C_q-quasi-greedy and C_a-SLC, then the basis is C_{al}-almost-greedy with
$$C_{al} \le C_q C_a.$$

Proof. a) Assume that $\mathscr{B}$ is C_{al}-almost-greedy. Since
$$\|x - \mathscr{G}_m(x)\| \le C_{al} \inf\{\|x - P_A(x)\| : A \subset \mathbb{N}, |A| \le m\},$$
we can select $A = \emptyset$. Then, we obtain that $\|x - \mathscr{G}_m(x)\| \le C_{al}\|x\|$, so the basis is quasi-greedy with constant $C_q \le C_{al}$.

Respect to the SLC, the same proof we gave in Theorem 2.3.7 actually gives $C_a \le C_{al}$.

b) Assume that $\mathscr{B}$ is C_{sd}-disjoint-super-democratic and C_q-quasi-greedy. For $x \in \mathbb{X}$, let $\mathscr{G}_m(x) = P_A(x)$ be the m-th greedy sum of x. For each $\varepsilon > 0$, find B such that $\|x - P_B(x)\| < \widetilde{\sigma}_m(x) + \varepsilon$, with $|B| \le m$. We have the following decomposition:
$$x - \mathscr{G}_m(x) = P_{(A\cup B)^c}(x - P_B(x)) + P_{B\setminus A}(x).$$
Of course, since $A \setminus B$ is a greedy set of $x - P_B(x)$ for $k := |A \setminus B| \in \mathbb{N}$,
$$\|P_{(A\cup B)^c}(x - P_B(x))\| \le C_q \|x - P_B(x)\|, \tag{2.52}$$

so we only have to estimate the quantity $\|P_{B\setminus A}(x)\|$. For that, arguing with the same technique than in the item b) of Theorem 2.3.7, taking $\varepsilon' \equiv \{\mathrm{sign}(\mathbf{e}_j^*(x))\}$,

$$\|P_{B\setminus A}(x)\| \leq C_{sd} \min_{j\in A\setminus B} |\mathbf{e}_j^*(x)|\|\mathbf{1}_{\varepsilon'(A\setminus B)}\|. \tag{2.53}$$

Now, since $A\setminus B$ is greedy for $x - P_B(x)$, using the Proposition 2.1.14 in (2.53),

$$\min_{j\in A\setminus B} |\mathbf{e}_j^*(x)|\|\mathbf{1}_{\varepsilon' A\setminus B}\| \leq 2C_q\|x - P_B(x)\|. \tag{2.54}$$

Thus, by (2.52), (2.53) and (2.54), $\mathscr{B}$ is almost-greedy with $C_{al} \leq C_q(1+2C_{sd})$.

c) Now, take x, A and B as in b). Consider $t := \min\{|\mathbf{e}_n^*(x)| : n \in A\}$ and $\eta \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$. Arguing in the same way than in the item c) of Theorem 2.3.7,

$$\begin{aligned}
\|x - \mathscr{G}_m(x)\| &\leq C_a\|P_{(A\cup B)^c}(x - P_B(x)) + t\mathbf{1}_{\eta(A\setminus B)}\| \\
&= C_a\|T_t((\mathrm{Id}_{\mathbb{X}} - P_B)(x))\| \\
&\leq C_qC_a\|x - P_B(x)\|.
\end{aligned}$$

This gives the desired result. □

Remark 2.5.5. Notice that the item c) of the above theorem was proved by F. Albiac and J. L. Ansorena in [4] using similar techniques.

Example 2.5.6. One example of almost-greedy basis and not greedy is the Example 2.1.9. Indeed, to show that the basis is democratic, consider a set $A \subset \mathbb{N}$ with $|A| = m$. We have that

$$\|\mathbf{1}_A\|_2 = \sqrt{m},$$

and

$$\sum_{n\in A} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{m} \frac{1}{\sqrt{n}} \leq \int_0^m \frac{dx}{\sqrt{x}} = 2\sqrt{m}.$$

Hence, for all finite set A, $\|\mathbf{1}_A\| \approx \sqrt{|A|}$, so the basis is democratic.

2.6 Partially-greedy bases

In practice, one would like to know whether the greedy approximation $\mathscr{G}_m(x)$ always performs better than the standard linear approximation $P_m(x)$. To this end, S. J. Dilworth et al. ([32]) introduced the notion of partially greedy basis. Namely, an SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ was called partially greedy if

$$\|x - \mathscr{G}_m(x)\| \leq C\|x - P_m(x)\|,\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{2.55}$$

When $\mathscr{B}$ is a Schauder basis (which is the setting considered in [32]), this notion is equivalent to the slightly stronger assertion

$$\|x - \mathscr{G}_m(x)\| \leq C \inf_{k\leq m} \|x - P_k(x)\|. \tag{2.56}$$

In [32], it is shown that a Schauder basis $\mathscr{B}$ is partially greedy if and only if the basis is quasi-greedy and conservative. Motivated by (2.56), we give the following definition.

Definition 2.6.1. An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *partially-greedy* if there exists a positive constant C such that

$$\|x-\mathscr{G}_m(x)\| \leq C \inf_{k\leq m} \|x-P_k(x)\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{2.57}$$

The least constant C in (2.57) is denoted by $C_p[\mathscr{B},\mathbb{X}] := C_p$ and we will say that $\mathscr{B}$ is C_p-*partially-greedy*.

The next theorem, which is inspired by [21], generalizes the results in [32] and additionally removes the Schauder basis condition.

Theorem 2.6.2. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_p-partially-greedy, then $\mathscr{B}$ is C_{sc}-super-conservative and C_q-quasi-greedy with

$$\max\{C_{sc}, C_q\} \leq C_p.$$

b) If $\mathscr{B}$ is C_q-quasi-greedy and C_{sc}-super-conservative, $\mathscr{B}$ is C_p-partially-greedy with constant

$$C_p \leq 1 + 2C_q + 2C_q C_{sc}.$$

Proof. a) Assume that $\mathscr{B}$ is C_p-partially-greedy. To show that $\mathscr{B}$ is super-conservative, take $(A,B,\varepsilon,\varepsilon') \in \mathfrak{F}_c'$ and show (2.19). Let $m = \max A$ and define the set $D = [1,..,m] \setminus A$. Of course,

$$m = |A \cup D| \leq |B \cup D|.$$

Define now $x := \mathbf{1}_{\varepsilon A} + (1+\delta)\mathbf{1}_{\varepsilon'(B\cup D)}$. Then,

$$\|\mathbf{1}_{\varepsilon A}\| = \|x - \mathscr{G}_{|B\cup D|}(x)\| \leq C_p \|x - P_m(x)\| = C_p \|(1+\delta)\mathbf{1}_{\varepsilon' B}\|.$$

Taking $\delta \searrow 0$, the basis is C_{sc}-super-conservative with $C_{sc} \leq C_p$.

To prove the quasi-greediness, since we know that

$$\|x-\mathscr{G}_m(x)\| \leq C \inf_{k\leq m} \|x-P_k(x)\|,$$

taking $k=0$, the basis is quasi-greedy with $C_q \leq C_p$.

b) Now, assume that $\mathscr{B}$ is C_{sc}-super-conservative and C_q-quasi-greedy, and show that $\mathscr{B}$ is partially-greedy. Take $x \in \mathbb{X}$, $\mathscr{G}_m(x)$ the m-th greedy sum of x, and $k \leq m$. Define the following sets:

$$D := \{\rho(j) : j \leq m, \rho(j) \leq k\}, \quad B := \{\rho(j) : j \leq m, \rho(j) > k\}, \quad A := [1,...,k] \setminus D,$$

where ρ is the natural greedy ordering. Then $|A| = k - |D| \leq m - |D| = |B|$. For the element x, we have the following decomposition using these sets:

$$x - \mathscr{G}_m(x) = (x - P_k(x)) - P_B(x) + P_A(x).$$

On the one hand, $\|P_B(x)\| = \|G_n(x - P_k(x))\| \leq 2C_q\|x - P_k(x)\|$ for $n := |B|$. On the other hand, by (i) of Corollary 1.3.3 and super-conservativeness with $|A| \leq |B|$,

$$\|P_A(x)\| \leq \max_{i\in A} |\mathbf{e}_i^*(x)| \sup_{\varepsilon\in\Psi_A} \|\mathbf{1}_{\varepsilon A}\| \leq C_{sc} \max_{i\in A} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\|,$$

for any $\eta \in \Psi_B$. Now, using Lemma 2.1.14 with $\eta \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$,

$$\begin{aligned} \|P_A(x)\| &\leq C_{sc} \max_{i\in A} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\| \leq C_{sc} \min_{i\in B} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\| \\ &= C_{sc} \min_{i\in B} |\mathbf{e}_i^*(x - P_m(x))| \, \|\mathbf{1}_{\eta B}\| \\ &\leq 2C_q C_{sc} \|x - P_m(x)\|. \end{aligned}$$

Then, $\|x - \mathscr{G}_m(x)\| \leq (1 + 2C_q + 2C_q C_{sc})\|x - P_k(x)\|$, for any $k \leq m$. □

Remark 2.6.3. In [32], the authors showed that $C_p = O(C_q^4 C_c)$, where C_c is the conservative constant. It is easy to see that $C_{sc} \lesssim C_q C_c$, so our Theorem 2.6.2 improves the bound of C_p since we obtain that $C_p = O(C_q C_{sc}) = O(C_q^2 C_c)$.

Example 2.6.4. We present an example of a partially-greedy basis that is not almost-greedy. This example can be found in [21]. Define the set

$$\mathscr{S} = \{A \in \mathbb{N}^{<\infty} : |A| \leq \sqrt{\min A}\}.$$

Observe that $\mathscr{S}$ has the *spreading property*, i.e, if $m \in \mathbb{N}$, $\{f_1, ..., f_m\} \in \mathscr{S}$ and $\{g_1, ..., g_m\} \in \mathbb{N}^m$ with $f_i \leq g_i$ for all $i = 1, \ldots, m$, then $\{g_1, ..., g_m\} \in \mathscr{S}$. It is also *hereditary*, i.e., if $A \in \mathscr{S}$ and $B \subset A$ then $B \in \mathscr{S}$.

Now, consider a slight modification of the Schreier space (see [70]). Let $\mathbb{X}$ be the Banach space defined as the completion of c_{00} under the norm

$$\|(a_n)_n\| = \sup_{A\in\mathscr{S}} \sum_{n\in A} |a_n|.$$

Of course, the canonical basis $(\mathbf{e}_n)_n$ is a normalized 1-unconditional basis. Note that the hereditary property guarantees that for any $\varepsilon \in \Psi_A$,

$$\|\mathbf{1}_{\varepsilon A}\| = \sup_{F\in\mathscr{S}, F\subseteq A} |F|.$$

Now, if $A < B$ and $|A| \leq |B|$, then there is $F \in \mathscr{S}$ with $F \subseteq A$ such that $\|\mathbf{1}_{\varepsilon A}\| = |F|$. By the spreading property, we can "push out" F to obtain a set $G \subseteq B$ such that $G \in \mathscr{S}$ and $|G| = |F|$. Hence, for any $\varepsilon \in \Psi_A$ and $\varepsilon' \in \Psi_B$,

$$\|\mathbf{1}_{\varepsilon A}\| = |F| = |G| \leq \|\mathbf{1}_{\varepsilon' B}\|.$$

Thus, the basis is super-conservative with constant $C_{sc} = 1$.

To prove that the basis is not democratic, we can select the sets $A = \{N^2 + 1, ..., N^2 + N\}$ and $B = \{1, ..., N\}$. Then, since $A \in \mathscr{S}$, $\|\mathbf{1}_A\| = N$. However, $\|\mathbf{1}_B\| \leq \sqrt{N}$: take a set $A_1 \in \mathscr{S}$ such that $A_1 \subset B$ and $\|\mathbf{1}_B\| = |A_1|$. Then, $\min A_1 \leq N$, so $|A_1| \leq \sqrt{N}$. Hence, $\|\mathbf{1}_B\| \leq \sqrt{N}$ and the basis is not democratic.

2.7 Open questions

We pose the following open questions related to greedy bases:

- We have seen in Theorem 2.5.4 that using the symmetry for largest coefficients and quasi-greediness we can recover the constant $C_{al} = 1$, but, F. Albiac and J. L. Ansorena proved that the condition of 1-symmetry for largest coefficients is necessary and sufficient condition to characterize 1-almost-greediness. The result is the following:

  Theorem 2.7.1 ([3]). An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is 1-almost-greedy if and only if $\mathscr{B}$ is 1-symmetric for largest coefficients. In particular, if $\mathscr{B}$ is 1-symmetric for largest coefficients, $\mathscr{B}$ is 1-quasi-greedy.

  Question 1: would it be the 1-symmetry for largest coefficients necessary and sufficient condition to characterize the 1-greediness? In other words, it is possible to prove that if a basis $\mathscr{B}$ is 1-symmetric for largest coefficients then $\mathscr{B}$ is C-unconditional for some constant $C \geq 1$?

- We have characterized partially-greedy bases under quasi-greediness and conservativeness and, of course, under the condition of Schauder bases, we recover the definition of partially-greediness introduced by Dilworth et al. (2.55).

  Question 2: if $\mathscr{B}$ is an SM-basis in a Banach space and satisfies (2.55), is there a positive constant C such that

$$\|x - \mathscr{G}_m(x)\| \leq C \inf\{\|x - P_k(x)\| : k \leq m\}, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}?$$

Chapter 3

The Haar system and Bidemocratic bases

The Haar system $\mathscr{H}^{(p)}$ in the spaces $L_p([0,1))$ (L_p for short) is a very important basis in many fields of mathematics. To define $\mathscr{H}^{(p)}$, we use the following notation: $\mathscr{D}$ is the collection of all dyadic intervals contained in $[0,1)$, $\mathscr{D}_0$ denotes the set $\mathscr{D}\cup\{0\}$, $h_0^{(p)}$ is the constant function 1 on $[0,1)$, and $h_I^{(p)}$ stands for the L_p-normalized Haar function supported on I, i.e., if $I=[(j-1)2^{-n}, j2^{-n})$ with $j=1,...,2^n$, $n=0,1,...$, then

$$h_I^{(p)}(t)=\begin{cases} 2^{n/p} & \text{if } t\in\left[(j-1)2^{-n},(j-\frac{1}{2})2^{-n}\right), \\ \\ -2^{n/p} & \text{if } t\in\left[(j-\frac{1}{2})2^{-n},j2^{-n}\right). \end{cases}$$

We define $\mathscr{H}^{(p)}=(h_I^{(p)})_{I\in\mathscr{D}_0}$. It is well known (see [10, Proposition 6.1.3]) that $\mathscr{H}^{(p)}$ is a Schauder basis for L_p when $1\le p<\infty$, whereas $\mathscr{H}^{(\infty)}$ is a basis for its closed span of $\mathscr{H}^{(p)}$ in L^∞. In either case, the family of coordinate functionals of $\mathscr{H}^{(p)}$ is $\mathscr{H}^{(p')}$ (with the canonical isometric identification of functions in $L_{p'}$ with functionals in $(L_p)^*$, where $p'=p/(p-1)$ and $1\le p<\infty$).

V. N. Temlyakov ([77]) showed that $\mathscr{H}^{(p)}$ is a greedy basis in L_p for $1<p<\infty$. Note that neither $\mathscr{H}^{(1)}$ is a greedy basis for L_1 nor $\mathscr{H}^{(\infty)}$ is a greedy basis for the closed span of $\mathscr{H}^{(\infty)}$ in L^∞. Indeed, it can be seen that the Haar system is not unconditional in L_1 and in L^∞, hence, is not greedy in theses spaces using Theorem 2.3.7. Therefore, we may expect that

$$\lim_{p\to 1^+} C_g[\mathscr{H}^{(p)},L_p]=\infty=\lim_{p\to\infty} C_g[\mathscr{H}^{(p)},L_p].$$

The Haar system $\mathscr{H}^{(2)}$ is an orthonormal basis for L_2, hence $C_g[\mathscr{H}^{(2)},L_2]=1$. However, for $p\neq 2$ it quite hard to attempt to compute the exact value of $C_g[\mathscr{H}^{(p)},L_p]$. It is therefore natural to address the problem of obtaining asymptotic estimates for $C_g[\mathscr{H}^{(p)},L_p]$ as p tends to 1 or to ∞. In this chapter we will study and discuss this problem using the notion of bidemocratic bases introduced by S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov in [32]. The results that we present are in the papers [7, 17].

3.1 Bidemocratic bases

Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space $\mathbb{X}$. We first make the observation that the system $\mathscr{B}^* = (\mathbf{e}_n^*)_{n=1}^{\infty}$ may not be complete in $\mathbb{X}^*$, so, denoting by $\mathbb{Y} = [\mathscr{B}^*]$, we consider in this subspace $\mathbb{Y}$ the fundamental function of the dual space $\varphi_u^*(m)$, where

$$\varphi_u^*[\mathscr{B}^*, \mathbb{Y}](m) := \varphi_u^*(m) = \sup_{\varepsilon \in \Psi_A, |A| \leq m} \|\mathbf{1}_{\varepsilon A}^*\|_* = \sup_{\varepsilon \in \Psi_A, |A| = m} \|\mathbf{1}_{\varepsilon A}^*\|_*,$$

where $\mathbf{1}_{\varepsilon A}^*[\mathscr{B}^*, \mathbb{Y}] := \mathbf{1}_{\varepsilon A}^* = \sum_{n \in A} \varepsilon_n \mathbf{e}_n^*$.

Bidemocratic bases arise in relation with the following natural question: if an SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is greedy, does this imply that $\mathscr{B}^*$ is greedy in $\mathbb{Y}$?

In general, the answer is false since if an SM-basis is democratic in $\mathbb{X}$, $\mathscr{B}^*$ may not be democratic in $\mathbb{Y}$ (see Lemma 4.3.5 below). S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov defined a notion to guarantee that greediness is preserved in the dual space: the bidemocracy ([32]).

Observe that for all finite set $A \subset \mathbb{N}$ of cardinality $m \in \mathbb{N}$ and $\varepsilon \in \Psi_A$, then

$$m = \mathbf{1}_{\varepsilon A}^*(\mathbf{1}_{\overline{\varepsilon} A}) \leq \varphi_u(m) \varphi_u^*(m),$$

where $\overline{\varepsilon}$ is the conjugate of ε.

Definition 3.1.1 ([32])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *bidemocratic* if there is a positive constant C such that

$$\varphi_u(m) \varphi_u^*(m) \leq Cm, \; \forall m \in \mathbb{N}. \tag{3.1}$$

The least constant that verifies (3.1) is denoted by $C_b[\mathscr{B}, \mathbb{X}] := C_b$ and we will say that $\mathscr{B}$ is *C_b-bidemocratic*.

Theorem 3.1.2 ([10, Theorem 10.6.2])**.** Assume that $\mathscr{B}$ is an SM-unconditional basis in a Banach space $\mathbb{X}$. The following are equivalent:

- $\mathscr{B}$ is bidemocratic.
- $\mathscr{B}$ and $\mathscr{B}^*$ are both democratic.

Theorem 3.1.3 ([32, Theorem 5.4])**.** Assume that $\mathscr{B}$ is an SM-quasi-greedy basis in a Banach space $\mathbb{X}$. The following are equivalent:

- $\mathscr{B}$ bidemocratic.
- $\mathscr{B}$ and $\mathscr{B}^*$ are both almost-greedy.

In Chapter 4 (see Subsection 4.3.2), we present an example of an almost-greedy basis $\mathscr{B}$ that is not bidemocratic because $\mathscr{B}^*$ is not democratic. The details are carried out in the paper [19]. We study now some consequences of bidemocracy that can be found, for instance, in [7, 21, 32].

Proposition 3.1.4. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space $\mathbb{X}$. Assume that $\mathscr{B}$ is C_b-bidemocratic. Then,

a) $\mathscr{B}$ and $\mathscr{B}^*$ are both super-democratic with constant $\max\{C_s, C_s^*\} \le C_b$, where $C_s[\mathscr{B}^*, \mathbb{Y}] := C_s^*$.

b) For every $x \in \mathbb{X}$, and every finite set $A \subset \operatorname{supp}(x)$,

$$\min_{j\in A} |\mathbf{e}_j^*(x)|\varphi_u(|A|) \le C_b\|x\|. \tag{3.2}$$

c) $\mathscr{B}$ is C_a-symmetric for largest coefficients with $C_a \le 1+2C_b$.

d) For every finite set $A \subset \mathbb{N}$ and $x \in \mathbb{X}$,

$$\|P_A(x)\| \le C_b \frac{\max\{|\mathbf{e}_j^*(x)| : j \in A\}}{\min\{|\mathbf{e}_j^*(x)| : j \in A\}}\|x\|.$$

Moreover, as a corollary, $\|P_A\| \lesssim \ln(m+1)$ for every set $A \in \mathbb{N}^m$.

Proof. a) This proof can be found in [32]. Indeed, if we take $A, B \in \mathbb{N}^{<\infty}$ such that $|A| \le |B|$ and $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$,

$$\|\mathbf{1}_{\varepsilon A}\| \le \varphi_u(|A|) \le \varphi_u(|B|) \le C_b \frac{|B|}{\|\mathbf{1}^*_{\overline{\varepsilon' B}}\|_*} = C_b \frac{\mathbf{1}^*_{\overline{\varepsilon' B}}(\mathbf{1}_{\varepsilon' B})}{\|\mathbf{1}^*_{\overline{\varepsilon' B}}\|_*} \le C_b \frac{\|\mathbf{1}_{\varepsilon' B}\|\|\mathbf{1}^*_{\overline{\varepsilon' B}}\|_*}{\|\mathbf{1}^*_{\overline{\varepsilon' B}}\|_*} = C_b\|\mathbf{1}_{\varepsilon' B}\|.$$

Hence, $\mathscr{B}$ is super-democratic. To prove that $\mathscr{B}^*$ is also super-democratic, we can use a similar argument.

b) Take $x \in \mathbb{X}$, a finite set $A \subset \operatorname{supp}(x)$ and the sign $\varepsilon \equiv \{\overline{\operatorname{sign}(\mathbf{e}_j^*(x))}\}_{j\in A}$. Then, if we denote by $\alpha = \min_{j\in A}|\mathbf{e}_j^*(x)|$,

$$\alpha\varphi_u(m) \le C_b \frac{\alpha|A|}{\|\mathbf{1}^*_{\varepsilon A}\|_*} \le C_b \frac{\sum_{j\in A}|\mathbf{e}_j^*(x)|}{\|\mathbf{1}^*_{\varepsilon A}\|_*} = C_b \frac{\mathbf{1}^*_{\varepsilon A}(x)}{\|\mathbf{1}^*_{\varepsilon A}\|_*} \le C_b\|x\|.$$

c) Take $x \in \mathbb{X}$, $A, B \in \mathbb{N}^{<\infty}$ with $|A| \le |B|$, $A \cap B = \emptyset$, $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$ and $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$.

$$\|x + \mathbf{1}_{\varepsilon A}\| \le \|x + \mathbf{1}_{\varepsilon' B}\| + \|\mathbf{1}_{\varepsilon' B}\| + \|\mathbf{1}_{\varepsilon A}\|. \tag{3.3}$$

Define now $y := x + \mathbf{1}_{\varepsilon' B}$. Since $\min_{n\in B}|\mathbf{e}_n^*(y)| = 1$, using b),

$$\max\{\|\mathbf{1}_{\varepsilon A}\|, \|\mathbf{1}_{\varepsilon' B}\|\} \le \min_{n\in B}|\mathbf{e}_n^*(y)|\varphi_u(|B|) \le C_b\|y\| = C_b\|x + \mathbf{1}_{\varepsilon' B}\|.$$

Hence, $\|x + \mathbf{1}_{\varepsilon A}\| \le (1+2C_b)\|x + \mathbf{1}_{\varepsilon' B}\|$.

d) Take $x \in \mathbb{X}$ and $A \subset \operatorname{supp}(x)$ finite. Since $\|\mathbf{1}_{\varepsilon A}\| \le \varphi_u(|A|)$, using Corollary 1.3.3 and b),

$$\begin{aligned}\|P_A(x)\| \le \max_{j\in A}|\mathbf{e}_j^*(x)|\varphi_u(m) &= \frac{\max_{j\in A}|\mathbf{e}_j^*(x)|}{\min_{j\in A}|\mathbf{e}_j^*(x)|}\min_{j\in A}|\mathbf{e}_j^*(x)|\varphi_u(m)\\ &\le C_b\frac{\max_{j\in A}|\mathbf{e}_j^*(x)|}{\min_{j\in A}|\mathbf{e}_j^*(x)|}\|x\|.\end{aligned}$$

That establishes the first part of e). To prove the second one, take $m \geq 2$ and let k be such that $k = \lfloor \log_2(m) \rfloor$, that is, $2^k \leq m < 2^{k+1}$. Let $x \in \mathbb{X}$ with $\|x\| = 1$. We consider the following sets: denote by $B_0 = \{n \in \mathbb{N} : |\mathbf{e}_n^*(x)| \leq \mathbf{c}^* 2^{-k}\}$, and for $1 \leq j \leq k$, let

$$B_j = \{n \in \mathbb{N} : \mathbf{c}^* 2^{-j} < |\mathbf{e}_n^*(x)| \leq \mathbf{c}^* 2^{-j+1}\}.$$

Note that $(B_j)_{j=0}^k$ forms a partition of $\mathbb{N}$. If we consider any set $A \subset \operatorname{supp}(x)$ with $|A| = m$, using the first part of this item, $\|P_{A \cap B_j}(x)\| \leq 2C_b$ for $j = 1, ..., k$. On the other hand, $\|P_{A \cap B_0}(x)\| \leq \mathbf{c}\mathbf{c}^* 2^{-k} |A \cap B_0| \leq 2\mathbf{c}\mathbf{c}^* 2^{-k-1} m \leq 2\mathbf{c}\mathbf{c}^*$. Hence,

$$\|P_A(x)\| = \left\| \sum_{j=0}^{k} P_{A \cap B_j}(x) \right\| \leq \sum_{j=0}^{k} \|P_{A \cap B_j}(x)\| \leq 2kC_b + 2\mathbf{c}\mathbf{c}^* \lesssim C_b \ln(m+1) \|x\|.$$

□

Remark 3.1.5. Item d) of the last result shows that if a basis is bidemocratic, we can control the unconditionality by a logarithm factor and, in fact, this factor is sharp (see Subsection 4.3.5 of Chapter 4). Also, d) is shared by all quasi-greedy bases. It is then natural to ask whether bidemocracy implies quasi-greediness. This seems to be an open question in the field.

We finish this section with a characterization of bidemocracy under the condition URP, where we will use some ideas of Proposition 4.4 of [32]

Definition 3.1.6. An SM-basis $\mathscr{B}$ in a Banach space has the *Upper Regularity Property* (URP) if there exists $\beta \in (0,1)$ and $C_\beta \geq 1$ such that

$$\varphi_u(m) \leq C_\beta \left(\frac{m}{k}\right)^\beta \varphi_u(k), \ \forall m \geq k.$$

Example 3.1.7. If for an SM-basis $\mathscr{B}$, $\varphi_u(m) \approx m^\alpha$ with $\alpha \in [0,1)$, $\mathscr{B}$ has the URP. For instance, if $\mathscr{B}$ is the canonical basis in the space ℓ_p, with $1 < p < \infty$, $\mathscr{B}$ has the URP. However, if $\mathscr{B}$ is the canonical basis of ℓ_1, $\mathscr{B}$ has not the URP.

Theorem 3.1.8. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$ with the URP. Then, $\mathscr{B}$ is bidemocratic if and only if (3.2) is satisfied.

Proof. Thanks to Proposition 3.1.4, it is only necessary to show that (3.2) implies bidemocracy. Assume that (3.2) holds with constant C. Take $\varepsilon \in \Psi_A$ and $A \subset \mathbb{N}$ of cardinality m. Take now $x \in \mathbb{X}$ with $\|x\| = 1$ such that $\frac{1}{2}\|\mathbf{1}_{\varepsilon A}^*\|_* < \sum_{j \in A} |\mathbf{e}_j^*(x)|$. Note that, if ρ is the natural greedy ordering for x, using (3.2) with constant C, for all $j \in \mathbb{N}$,

$$|\mathbf{e}_{\rho(j)}^*(x)| \leq C \frac{\|x\|}{\varphi_u(j)} = C \frac{1}{\varphi_u(j)}.$$

Using this fact,

$$\begin{aligned} \varphi_u(m) \|\mathbf{1}_{\varepsilon A}^*\|_* &\leq 2 \sum_{j \in A} |\mathbf{e}_j^*(x)| \varphi_u(m) \leq 2 \sum_{j=1}^{m} |\mathbf{e}_{\rho(j)}^*(x)| \varphi_u(m) \\ &\leq 2C \sum_{j=1}^{m} \frac{\varphi_u(m)}{\varphi_u(j)} \leq 2CC_\beta \sum_{j=1}^{m} \frac{m^\beta}{j^\beta} \leq \frac{2CC_\beta m^\beta}{(1-\beta)} m^{1-\beta} = \frac{2CC_\beta}{1-\beta} m. \end{aligned}$$

Hence, the basis is bidemocratic with constant $C_b \leq \dfrac{2CC_\beta}{1-\beta}$. □

Example 3.1.9. If $\mathscr{B}$ is an SM-almost-greedy basis, (3.2) is satisfied. To show that, we only have to know that almost-greediness is equivalent to super-democracy and quasi-greediness (see Theorem 2.5.4). Assume that $\mathscr{B}$ is C_s-super-democratic and C_q-quasi-greedy. Let $A \in \mathbb{N}^m$ and $x \in \mathbb{X}$ with Λ a greedy set of x with cardinality $|\Lambda| = m$. If $\varepsilon \equiv \{\mathrm{sign}(\mathbf{e}_j^*(x))\}$, using the super-democracy and Proposition 2.1.14,

$$\min_{j\in A} |\mathbf{e}_j^*(x)|\varphi_u(|A|) \leq C_s \min_{j\in\Lambda} |\mathbf{e}_j^*(x)| \|\mathbf{1}_{\varepsilon\Lambda}\| \leq 2C_q C_s \|x\|.$$

One almost-greedy basis without the URP condition and not bidemocratic is the Lindestrauss basis (see Subsection 4.3.2).

Finally, we present a new proposition that can be found in [7] that shows that bidemocracy and unconditionality imply greediness with an additive bound of C_g.

Proposition 3.1.10. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. If $\mathscr{B}$ is K_{su}-unconditional and C_b-bidemocratic, then $\mathscr{B}$ is C_g-greedy with $C_g \leq K_{su} + C_b$.

Proof. Let $x \in \mathbb{X}$ and $A = \mathrm{supp}(\mathscr{G}_m(x))$. For each $\varepsilon > 0$, find $y = \sum_{n\in B} z_n \mathbf{e}_n$ with $|B| \leq m$ and $\|x-y\| < \sigma_m(x) + \varepsilon$. As in the proof of Theorem 2.3.7,

$$x - \mathscr{G}_m(x) = P_{(A\cup B)^c}(x-y) + P_{B\setminus A}(x).$$

Since

$$\|P_{(A\cup B)^c}(x-y)\| \leq K_{su}\|x-y\|, \tag{3.4}$$

we only have to estimate $\|P_{A\setminus B}(x)\|$. First, denoting by $k := |A\setminus B|$, we apply Corollary 1.3.3 obtaining that

$$\|P_{B\setminus A}(x)\| \quad \leq \quad \max_{n\in B\setminus A} |\mathbf{e}_n^*(x)|\varphi_u(k). \tag{3.5}$$

Now, applying b) of Proposition 3.1.4 in (3.5),

$$\|P_{B\setminus A}(x)\| \quad \leq \quad \max_{n\in B\setminus A} |\mathbf{e}_n^*(x)|\varphi_u(k) \leq \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-y)|\varphi_u(k) \leq C_b\|x-y\|.$$

Hence, by (3.4) and (3.5), we obtain that $\mathscr{B}$ is greedy with $C_g \leq K_{su} + C_b$. □

3.2 The Haar system in $L_p([0,1))$ and its greedy constant

Below we shall use the following result that gives the sharp unconditional constant for the Haar basis. Recall that K_u is defined in (1.4).

Theorem 3.2.1 ([25]). If $1 < p < \infty$, then $K_u[\mathscr{H}^{(p)}, L_p] = p^* - 1$, where $p^* = \max\{p, p'\}$ and p' is the conjugate of p.

As $K_{su} \le K_u \le 2\kappa K_{su}$ (see (1.5)), using Theorem 2.3.7, we infer that $\frac{p^*-1}{2\kappa} \le C_g[\mathscr{H}^{(p)}, L_p]$. For the upper bound, using Proposition 3.1.10, we only need to know an upper bound of $C_b[\mathscr{H}^{(p)}, L_p]$ for $1 < p < \infty$. We give this bound and also we will prove that democracy and super-democracy constants are also like p^*. These results are in [7].

Proposition 3.2.2. If $1 < p < \infty$ then

$$C_b[\mathscr{H}^{(p)}, L_p] \le D_p := \frac{8}{(2^{1/p}-1)(2^{1/p'}-1)}. \tag{3.6}$$

Proof. For $I \in \mathscr{D}$ let $n(I)$ be such that $|I| = 2^{-n(I)}$. Let $A \subseteq \mathscr{D}$ finite and $\varepsilon = (\varepsilon_I)_{I \in A}$ be such that $|\varepsilon_I| = 1$ for all $I \in A$. For $J \in A$ set

$$R_J = J \setminus \cup\{I \colon I \in A, n(I) > n(J)\}.$$

Taking into account that, for $n \in \mathbb{N}$, the collection of dyadic intervals $\{I \in \mathscr{D} \colon n(I) = n\}$ is a partition on $[0,1)$ we infer that

- $(R_I)_{I \in A}$ is a partition of $K = \cup_{I \in A} I$,
- $R_I \subseteq I$ for every $I \in A$, and
- given $t \in K$ and $k \in \mathbb{N}$ there is at most one interval $I_{t,k} \in A \cap \mathscr{D}_k$ such that $t \in I_{t,k}$; moreover $n(I_{t,k}) \le n(J)$ if $t \in J$.

Consequently, for any $t \in K$,

$$\begin{aligned}
\left| \sum_{I \in A} \varepsilon_I h_I^{(p)}(t) \right| &\le \sum_{I \in A} \left| h_I^{(p)}(t) \right| \\
&= \sum_{I \in A} 2^{n(I)/p} \chi_I(t) \\
&\le \sum_{J \in A} \left(\sum_{n=-\infty}^{n(J)} 2^{n/p} \right) \chi_{R_J}(t) \\
&= \frac{1}{1-2^{-1/p}} \sum_{J \in A} 2^{n(J)/p} \chi_{R_J}(t).
\end{aligned}$$

Hence, if we set $a_p = 1/(1-2^{-1/p})$ we obtain

$$\begin{aligned}
\left\| \sum_{I \in A} \varepsilon_I h_I^{(p)} \right\|_p &\le a_p \left(\textstyle\sum_{J \in A} 2^{n(J)} |R_J| \right)^{1/p} \\
&\le a_p \left(\textstyle\sum_{J \in A} 2^{n(J)} |J| \right)^{1/p} \\
&= a_p |A|^{1/p}.
\end{aligned}$$

Therefore,

$$\left\| h_0^{(p)} + \sum_{I \in A} \varepsilon_I h_I^{(p)} \right\|_p \le 1 + a_p |A|^{1/p}.$$

We infer that, for $m \in \mathbb{N}$,

$$\varphi_u[\mathscr{H}^{(p)}, L_p](m) \le \max\{a_p m^{1/p}, 1 + a_p (m-1)^{1/p}\} \le 2 a_p m^{1/p}.$$

The fact that $\varphi_u^*[\mathscr{H}^{(p)}, L_p](m) = \varphi_u[\mathscr{H}^{(p')}, L_{p'}](m)$ for $m \in \mathbb{N}$ yields

$$\frac{\varphi_u[\mathscr{H}^{(p)}, L_p](m)\, \varphi_u^*[\mathscr{H}^{(p)}, L_p](m)}{m} \le 4 a_p a_{p'} \frac{m^{1/p} m^{1/p'}}{m} = 4 a_p a_{p'} = D_p.$$

Thus

$$C_b[\mathscr{H}^{(p)}, L_p] \le D_p.$$

□

Proposition 3.2.3. If $1 < p < \infty$ then

$$d_p := \frac{2^{1/p^\#} - 1}{2^{1/p^*} - 1} \le C_d[\mathscr{H}^{(p)}, L_p],$$

where $p^\# = \min\{p, p'\}$.

Proof. If $(J_j)_{j=1}^m$ are disjointly supported intervals in $\mathscr{D}$ we have

$$\left\| \sum_{j=1}^m h_{J_j}^{(p)} \right\|_p = m^{1/p}. \tag{3.7}$$

Let $(I_j)_{j=1}^\infty$ be the sequence in $\mathscr{D}$ defined recursively as follows: $I_1 = [0,1)$ and I_{j+1} is the left half of I_j. Set $q = p'$. Since

$$\sum_{j=1}^m h_{I_j}^{(p)}(x) = \sum_{j=0}^{m-1} 2^{j/p} \chi_{[0,\frac{1}{2^m})}(x) + \sum_{j=1}^{m-1} \left(-2^{j/p} + \sum_{k=0}^{j-1} 2^{k/p} \right) \chi_{[\frac{1}{2^{j+1}}, \frac{1}{2^j})}(x) - \chi_{[0,1/2)}(x),$$

then

$$\begin{aligned}
\left\| \sum_{j=1}^m h_{I_j}^{(p)} \right\|_p^p &= 2^{-m} \left| \sum_{k=0}^{m-1} 2^{k/p} \right|^p + \sum_{j=1}^{m-1} 2^{-j-1} \left| 2^{j/p} - \sum_{k=0}^{j-1} 2^{k/p} \right|^p + \frac{1}{2} \\
&= \frac{(1 - 2^{-m/p})^p + \sum_{j=1}^{m-1} |2^{1/q} - 1 - 2^{-(j+1)/p}|^p + (1 - 2^{-1/p})^p}{(2^{1/p} - 1)^p} \\
&= \frac{\|f - g\|_{\ell_p}^p}{(2^{1/p} - 1)^p},
\end{aligned}$$

where

$$f = (1, \underbrace{2^{1/q} - 1, \dots, 2^{1/q} - 1}_{m-1 \text{ times}}, 1), \quad g = (2^{-m/p}, 2^{-2/p}, \dots, 2^{-j/p}, \dots, 2^{-m/p}, 2^{-1/p}).$$

We have $\|g\|_{\ell_p} = 1$ and $\|f\|_{\ell_p} = (2+(m-1)(2^{1/q}-1)^p)^{1/p}$. Hence, by Minkowski's inequality,

$$\frac{(2+(m-1)(2^{1/q}-1)^p)^{1/p}-1}{2^{1/p}-1} \leq \left\| \sum_{j=1}^{m} h_{I_j}^{(p)} \right\|_p \leq \frac{(2+(m-1)(2^{1/q}-1)^p)^{1/p}+1}{2^{1/p}-1}. \quad (3.8)$$

Comparing (3.7) with (3.8) and letting m tend to ∞ we get

$$C_d[\mathscr{H}^{(p)}, L_p] \geq \max\left\{ \frac{2^{1/q}-1}{2^{1/p}-1}, \frac{2^{1/p}-1}{2^{1/q}-1} \right\},$$

as desired. □

Theorem 3.2.4. For every $1 < p < \infty$,

$$C_d[\mathscr{H}^{(p)}, L_p] \approx C_s[\mathscr{H}^{(p)}, L_p] \approx C_b[\mathscr{H}^{(p)}, L_p] \approx p^*,$$

and

$$C_g[\mathscr{H}^{(p)}, L_p] \approx p^*.$$

Proof. Let D_p and d_p be as in Proposition 3.2.3 and Propostion 3.2.2. By these propositions, since

$$d_p \leq C_d[\mathscr{H}^{(p)}, L_p] \leq C_s[\mathscr{H}^{(p)}, L_p] \leq C_b[\mathscr{H}^{(p)}, L_p] \leq D_p,$$

we prove the first part of the result since we have that $d_p \approx D_p \approx p^*$. For the second part, using Proposition 3.1.10 and the fact that

$$C_d[\mathscr{H}^{(p)}, L_p] \leq C_g[\mathscr{H}^{(p)}, L_p] \leq K_{su}[\mathscr{H}^{(p)}, L_p] + C_b[\mathscr{H}^{(p)}, L_p],$$

we obtain that $C_g[\mathscr{H}^{(p)}, L_p] \approx p^*$. □

3.3 Greediness of the canonical system in $\mathfrak{f}^p(\omega)$

In this chapter we study some conditions on a weight ω to obtain that the Haar system in $L_p([0,1),\omega)$ is greedy. To do this, we shall use the characterization of $L_p([0,1),\omega)$, $1 < p < \infty$, in terms of the sequence space $\mathfrak{f}^p(\omega)$ and study under which conditions in ω, the normalized canonical basis in $\mathfrak{f}^p(\omega)$ is greedy and as a consequence, we recover that the Haar system is greedy in $L_p([0,1),\omega)$ when $\omega \in A_p^d$. For that, we use that ω satisfies the dyadic Carleson reverse condition.

Let $\omega : [0,1] \to \mathbb{R}^+$ be a measurable weight and we denote $\omega(I) = \int_I \omega(x)dx$ for any $I \in \mathscr{D}$. We take $h_I = h_I^{(2)}$. In the space $L_p(\omega) = L_p([0,1),\omega)$ we denote

$$\|f\|_{p,\omega} = \left(\int_0^1 |f(x)|^p \omega(x) dx \right)^{1/p},$$

and

$$c_I(f,p,\omega) := c_I(f)\|h_I\|_{p,\omega} = c_I(f) \frac{\omega(I)^{1/p}}{|I|^{1/2}},$$

where

$$c_I(f) := \langle f, h_I \rangle = \int_0^1 f(x) h_I(x) dx.$$

It is well known that $\mathscr{H}^{(2)}$ is an orthonormal basis in $L_2([0,1))$ and for $1 < p < \infty$, the Littlewood-Paley's Theory gives

$$c_p \left\| \left(\sum_I |c_I(f) h_I|^2 \right)^{1/2} \right\|_p \leq \|f\|_p \leq C_p \left\| \left(\sum_I |c_I(f) h_I|^2 \right)^{1/2} \right\|_p \tag{3.9}$$

which implies that $\mathscr{H}^{(2)}$ is an unconditional basis in $L_p([0,1])$.

Recall that ω is said to be a dyadic A_p-weight, where $1 < p < \infty$, (denoted $\omega \in A_p^d$) if

$$A_p^d(\omega) = \sup_{I \in \mathscr{D}} m_I(\omega) \Big(m_I(\omega^{-1/(p-1)}) \Big)^{p-1} < \infty, \tag{3.10}$$

where $m_I(\omega) = \frac{\omega(I)}{|I|}$.

As one may expect, Littlewood-Paley theory holds for weights in the dyadic A_p-class.

Theorem 3.3.1. (see [1,54] for the multidimensional case) Let $1 < p < \infty$. If $\omega \in A_p^d$ then

$$\|f\|_{p,\omega} \approx \left\| \left(\sum_I |c_I(f,p,\omega) \frac{h_I}{\|h_I\|_{p,\omega}}|^2 \right)^{1/2} \right\|_{p,\omega}. \tag{3.11}$$

In particular $(\frac{h_I}{\|h_I\|_{p,\omega}})_I$ is a normalized unconditional basis in $L_p(\omega)$.

The greediness of the Haar basis in $L_p(\omega)$ goes back to M. Izuki (see [54,55]) who showed that this holds for weights in the class A_p^d. For our purpose, we start introducing the space $\mathfrak{f}^p(\omega)$ and we shall use the ideas in these papers to show that the canonical basis is greedy in this space.

Definition 3.3.2. Let $\omega : [0,1] \to \mathbb{R}^+$ be a measurable weight and $1 \leq p < \infty$. For each finite set of dyadic intervals Λ we define

$$f_\Lambda = \sum_{I \in \Lambda} c_I(f) h_I = \sum_{I \in \Lambda} c_I(f,p,\omega) \frac{h_I}{\|h_I\|_{p,\omega}},$$

and write

$$\|f\|_{\mathfrak{f}^p(\omega)} = \left\| \left(\sum_{I \in \Lambda} |c_I(f,p,\omega) \frac{h_I}{\|h_I\|_{p,\omega}}|^2 \right)^{1/2} \right\|_{p,\omega}.$$

The closure of $span(f_\Lambda : |\Lambda| < \infty)$ under this norm will be denoted $\mathfrak{f}^p(\omega)$.

From the definition $(\frac{h_I}{\|h_I\|_{p,\omega}})_I$ is an unconditional basis in $\mathfrak{f}^p(\omega)$ with constant 1, and due to (3.11)

$$\mathfrak{f}^p(\omega) = L_p(\omega), \quad \text{whenever} \quad \omega \in A_p^d.$$

Also, our space $\mathfrak{f}^p(\omega)$ is the sequence Triebel-Lizorkin space $\mathfrak{f}^s_{p,q}(\omega)$ with $q = 2$ and $s = 0$. Our aim is to analyze conditions on the weight ω for the basis to be greedy in $\mathfrak{f}^p(\omega)$. For such a purpose we do not need the weight to belong to A_p^d. In fact analyzing the proof in [54, 55] one notices that only the dyadic reverse doubling condition (see [42, p. 141]) was used. Recall that a weight ω satisfies the *dyadic reverse doubling condition* (DRDC for short) if there exists $\delta < 1$ such that

$$\omega(I') \leq \delta\omega(I), \forall I, I' \in \mathscr{D} \text{ with } I' \subsetneq I. \tag{3.12}$$

Let us introduce certain weaker conditions.

Definition 3.3.3. Let $\alpha > 0$ and ω be a measurable weight. We shall say that ω satisfies *the dyadic reverse Carleson condition* (DRCC for short) of order α with constant $C > 0$ whenever

$$\sum_{I\in\mathscr{D}, J\subseteq I} \omega(I)^{-\alpha} \leq C\omega(J)^{-\alpha}, \forall J \in \mathscr{D}. \tag{3.13}$$

Definition 3.3.4. Let $\alpha > 0$ and two sequences $(w_I)_{I\in\mathscr{D}}$ and $(v_I)_{I\in\mathscr{D}}$ of positive real numbers. We say that the pair $\Big((w_I)_{I\in\mathscr{D}}, (v_I)_{I\in\mathscr{D}}\Big)$ satisfies *α-DRCC* with constant $C > 0$ whenever

$$\sum_{I\in\mathscr{D}, J\subseteq I} w_I^{-\alpha} \leq C v_J^{-\alpha}, \forall J \in \mathscr{D}. \tag{3.14}$$

Remark 3.3.5. (i) If $\omega \in \cup_{p>1} A_p^d$ then ω satisfies the dyadic reverse doubling condition (see [42, pg 141 and Chapter IV, Lemma 2.2]).

(ii) If ω satisfies the dyadic reverse doubling condition then ω satisfies the dyadic reverse Carleson condition of order α with constant $\frac{1}{1-\delta^\alpha}$ for any $\alpha > 0$.

Indeed,

$$\sum_{J\subseteq I} \omega(I)^{-\alpha} \leq \omega(J)^{-\alpha} + \omega(J)^{-\alpha} \sum_{m=1}^{\infty} \delta^{m\alpha} \leq \frac{1}{1-\delta^\alpha}\omega(J)^{-\alpha}.$$

(iii) If ω satisfies the dyadic reverse Carleson condition of order α and $w_I = \omega(I)$ for each $I \in \mathscr{D}$ then $\Big((w_I)_{I\in\mathscr{D}}, (w_I)_{I\in\mathscr{D}}\Big)$ satisfies α-DRCC.

In general, we have the following inclusions between theses classes:

$$\cup_{p>1} A_p^d \subset DRDC \subset \alpha\text{-}DRCC \subset \beta\text{-}DRCC,\ \forall \beta > \alpha > 0.$$

We need the following lemmas, where the ideas in the proofs are similar in [29] (see also [54, 55]).

Lemma 3.3.6. Let ω be a weight and $(v_I)_{I\in\mathscr{D}}$ be a sequence of positive real numbers such that $\Big((v_I)_{I\in\mathscr{D}}, (\omega(I))_{I\in\mathscr{D}}\Big)$ satisfies 1-DRCC with constant C. Then

$$\left(\sum_{I\in\Lambda} \frac{\omega(I)}{v_I}\right)^{1/p} \leq C \left\| \sum_{I\in\Lambda} \frac{h_I}{\|h_I\|_{p,\omega}} \right\|_{\mathfrak{f}^p(\omega)}, \forall 1 \leq p < \infty. \tag{3.15}$$

Proof. We first write

$$\left\| \sum_{I\in\Lambda} \frac{h_I}{\|h_I\|_{p,\omega}} \right\|_{\mathfrak{f}^p(\omega)} = \left(\int_0^1 (\sum_{I\in\Lambda} \omega(I)^{-2/p}\chi_I)^{p/2}\omega(x)dx\right)^{1/p}. \tag{3.16}$$

Let $I(x)$ denote the minimal dyadic interval in Λ with regard to the inclusion relation that contains x. Now we use that

$$\sum_{I\in\mathscr{D}, I(x)\subseteq I} v_I^{-1} \le C\omega(I(x))^{-1}$$

to conclude that

$$\begin{aligned}
\Big(\sum_{I\in\Lambda}\frac{\omega(I)}{v_I}\Big)^{1/p} &= \Big(\sum_{I\in\Lambda}\int_I v_I^{-1}\omega(x)dx\Big)^{1/p} = \Big(\int_0^1 (\sum_{I\in\Lambda} v_I^{-1}\chi_I(x))\omega(x)dx\Big)^{1/p}\\
&\le C\Big(\int_0^1 \omega(I(x))^{-1}\omega(x)dx\Big)^{1/p}\\
&\le C\Big(\int_0^1 (\sum_{I\in\Lambda}\omega(I)^{-2/p}\chi_I(x))^{p/2}\omega(x)dx\Big)^{1/p}\\
&= C\|\sum_{I\in\Lambda}\frac{h_I}{\|h_I\|_{p,\omega}}\|_{\mathfrak{f}^p(\omega)}.
\end{aligned}$$

The proof is complete. □

Lemma 3.3.7. Let $1<p<\infty$, ω be a weight and $(v_I)_{I\in\mathscr{D}}$ a sequence of positive real numbers. If the collection $\Big((\omega(I))_{I\in\mathscr{D}}, (v_I)_{I\in\mathscr{D}}\Big)$ satisfies $2/p$-DRCC with constant $C>0$ then

$$\Big\|\sum_{I\in\Lambda}\frac{h_I}{\|h_I\|_{p,\omega}}\Big\|_{\mathfrak{f}^p(\omega)} \le C\Big(\sum_{I\in\Lambda}\frac{\omega(I)}{v_I}\Big)^{1/p} \tag{3.17}$$

for all finite family Λ of dyadic intervals.

Proof. Let $E=\cup_{I\in\Lambda}I$ and $I(x)$ the minimal dyadic interval in Λ with regard to the inclusion relation that contains x. From (3.14) we have that

$$\sum_{I\in\Lambda}\omega(I)^{-2/p}\chi_I(x)\le C v_{I(x)}^{-2/p}, \quad x\in E. \tag{3.18}$$

Now denote for each $I\in\Lambda$, $\tilde{I}=\{x\in E : I(x)=I\}$. Clearly $\tilde{I}\subseteq I$ and $\cup_{I\in\Lambda}I=\cup_{I\in\Lambda}\tilde{I}$. Hence applying (3.16) and (3.18) we obtain

$$\begin{aligned}
\Big\|\sum_{I\in\Lambda}\frac{h_I}{\|h_I\|_{p,\omega}}\Big\|_{\mathfrak{f}^p(\omega)} &\le C\Big(\int_E v_{I(x)}^{-1}\omega(x)dx\Big)^{1/p} = C\Big(\int_{\cup_{I\in\Lambda}\tilde{I}} v_{I(x)}^{-1}\omega(x)dx\Big)^{1/p}\\
&\le C\Big(\sum_{I\in\Lambda}\int_{\tilde{I}} v_I^{-1}\omega(x)dx\Big)^{1/p} \le C\Big(\sum_{I\in\Lambda} v_I^{-1}\int_I \omega(x)dx\Big)^{1/p}\\
&= C(\sum_{I\in\Lambda}\frac{\omega(I)}{v_I})^{1/p}.
\end{aligned}$$

The proof is now complete.

□

Combining Remark 3.3.5 and Lemmas 3.3.6 and 3.3.7 we obtain the following corollary.

Corollary 3.3.8. Let $1 < p < \infty$ and ω satisfying the DRCC of order $\min\{1, 2/p\}$ then

$$\left\| \sum_{I \in \Lambda} \frac{h_I}{\|h_I\|_{p,\omega}} \right\|_{\mathfrak{f}^p(\omega)} \approx |\Lambda|^{1/p} \tag{3.19}$$

for all finite family Λ of dyadic intervals.

Corollary 3.3.9. Let $1 < p < \infty$. Then, using Theorem 2.3.7:

- If ω satisfies DRCC of order $\min\{1, 2/p\}$, then, $\left(\frac{h_I}{\|h_I\|_{p,\omega}} \right)$ is unconditional and democratic in $\mathfrak{f}^p(\omega)$. Hence, the basis is greedy in $\mathfrak{f}^p(\omega)$.
- If ω is in the class A_p^d, the Haar system is greedy in $L_p([0,1), \omega)$.

3.4 Open questions

Concerning the greedy constant $C_g[\mathscr{H}^{(p)}, L_p]$ for $1 < p < \infty$, we propose the following two questions:

Question 1: is $C_g[\mathscr{H}^{(p)}, L_p]$ decreasing on $(1,2]$ and increasing on $[2,\infty)$?

Question 2: is $C_g[\mathscr{H}^{(p)}, L_p]$ symmetric with respect to $p = 2$? That is, is $C_g[\mathscr{H}^{(p)}, L_p] = C_g[\mathscr{H}^{(p')}, L_{p'}]$ for $p \in (1, \infty)$?

Chapter 4

Lebesgue-type parameters for the TGA

From the definition of greedy bases, we know that the m-th greedy error is comparable to the error $\sigma_m(x)$ for all $x \in \mathbb{X}$ and $m \in \mathbb{N}$. In this section we are going to see what happens if these errors are not comparable. For that, we remind the following notation: given an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ of a Banach space $\mathbb{X}$, a *greedy set* for $x \in \mathbb{X}$ of order m, written $A \in \mathscr{G}(x,m)$, is a set of indices $A \subset \mathbb{N}$ such that $|A| = m$ and

$$\min_{n\in A} |\mathbf{e}_n^*(x)| \geq \max_{n\notin A} |\mathbf{e}_n^*(x)|.$$

A *greedy operator* of order m is any mapping $G_m : \mathbb{X} \to \mathbb{X}$ such that

$$x \in \mathbb{X} \longmapsto G_m(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n,$$

with $A \in \mathscr{G}(x,m)$. We write $\mathscr{G}_m$ for the set of all greedy operators of order m and $\mathscr{G} = \cup_{m\geq 1}\mathscr{G}_m$. By convention, we set $\mathscr{G}_0 = \{0\}$. Given $G, G' \in \mathscr{G}$ we shall write $G' < G$ whenever $G \in \mathscr{G}_m$ and $G' \in \mathscr{G}_n$ with $0 \leq n < m$ and $A' \subset A$ for all x.

To quantify the performance of greedy operators with respect to m-term approximations, we consider, for every $m = 1,2,3,\ldots$, the smallest numbers $\mathbf{L}_m = \mathbf{L}_m[\mathscr{B},\mathbb{X}]$ and $\widetilde{\mathbf{L}}_m = \widetilde{\mathbf{L}}_m[\mathscr{B},\mathbb{X}]$ such that

$$\|x - G_m(x)\| \leq \mathbf{L}_m\, \sigma_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\; G_m \in \mathscr{G}_m \tag{4.1}$$

and

$$\|x - G_m(x)\| \leq \widetilde{\mathbf{L}}_m\, \widetilde{\sigma}_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, G_m \in \mathscr{G}_m. \tag{4.2}$$

As in [80, Chapter 2], we say that (4.1) is a *Lebesgue-type inequality* for the greedy algorithm, and $\mathbf{L}_m$ is its associated Lebesgue-type parameter. Also, we know that $\mathbf{L}_m = O(1)$ if and only if $\mathscr{B}$ is democratic and unconditional (Theorem 2.3.7) and $\widetilde{\mathbf{L}}_m = O(1)$ if and only if $\mathscr{B}$ is democratic and quasi-greedy (Theorem 2.5.4).

Since 2011, Lebesgue-type inequalities have been studied in the class of quasi-greedy bases (see [35, 38, 44, 82]). For general bases, however, it is a difficult problem to find bounds for $\widetilde{\mathbf{L}}_m$ and $\mathbf{L}_m$ which are both asymptotically optimal and described in terms of reasonable quantities (such as the unconditionality, quasi-greedy and democracy parameters). We will study in this chapter this problem. Some quantities that depend on $\mathscr{B}$ and $\mathbb{X}$ and that we will need are the following:

- Unconditionality parameters:

$$k_m = \sup_{|A|\le m} \|P_A\|, \text{ and } k_m^c = \sup_{|A|\le m} \|\mathrm{Id}_{\mathbb{X}} - P_A\|.$$

Of course, $|k_m - k_m^c| \le 1$ for all $m \in \mathbb{N}$.

- Quasi-greedy parameters[1]:

$$g_m = \sup_{G\in \cup_{k\le m}\mathscr{G}_k} \|G\| \text{ and } g_m^c = \sup_{G\in \cup_{k\le m}\mathscr{G}_k} \|\mathrm{Id}_{\mathbb{X}} - G\|.$$

As in the unconditionality parameter, $|g_m - g_m^c| \le 1$ for all $m \in \mathbb{N}$.

We shall also use the following variants

$$\hat{g}_m = \min\{g_m, g_m^c\}, \text{ and } \tilde{g}_m = \sup_{\substack{G\in \cup_{k\le m}\mathscr{G}_k \\ G'<G}} \|G - G'\| \;.$$

- Democracy (and superdemocracy) parameter:

$$\mu_m = \sup_{|A|\le|B|\le m} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \quad \text{and} \quad \tilde{\mu}_m = \sup_{\substack{|A|\le|B|\le m \\ \varepsilon\in\Psi_A,\eta\in\Psi_B}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|},$$

and their counterparts for disjoint sets $A\cap B = \emptyset$, denoted μ_m^d and $\tilde{\mu}_m^d$.

- The parameter of the symmetry for largest coefficients:

$$\nu_m = \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A} + x\|}{\|\mathbf{1}_{\eta B} + x\|} \,:\, |A| \le |B| \le m,\, \varepsilon \in \Psi_A, \eta \in \Psi_B,\, |x|_\infty \le 1,\, A \mathbin{\dot\cup} B \mathbin{\dot\cup} x \right\},$$

where $|x|_\infty = \sup_n |\mathbf{e}_n^*(x)|$, and $A \mathbin{\dot\cup} B \mathbin{\dot\cup} x$ means that A, B and $\mathrm{supp}(x)$ are pairwise disjoint.

All these are natural quantities in the greedy literature as we have studied in Chapter 2, and quite often it is not hard to compute them explicitly. Moreover, $\mathscr{B}$ is unconditional if and only if $\sup_m k_m < \infty$, $\mathscr{B}$ is quasi-greedy if and only if $\sup_m g_m < \infty$, etc. Now, we present some basic lemmas connecting the above parameters and the truncation operator defined in Chapter 2. The original results that we present here are in [18, 19].

Lemma 4.0.1. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. For each $m \in \mathbb{N}$ we have

$$g_m \le \tilde{g}_m \le \min\{2\hat{g}_m,\, g_m g_m^c,\, k_m\}. \tag{4.3}$$

In particular, $\tilde{g}_m = g_m$ when $g_m^c = 1$.

Proof. $g_m \le \tilde{g}_m \le k_m$ is obvious by definition and $\tilde{g}_m \le 2\hat{g}_m$ follows easily from the triangle inequality. Finally, for each $G \in \cup_{k\le m}\mathscr{G}_k$ and $G' < G$ we can write $G(x) - G'(x) = \sum_{n\in A_x\setminus A'_x} \mathbf{e}_n^*(x)\mathbf{e}_n$

[1] We use the notation $\|G\| = \sup_{x\neq 0} \|Gx\| / \|x\|$, even if $G : \mathbb{X} \to \mathbb{X}$ may be a non-linear map.

with $A_x \setminus A'_x \in \cup_{k\le m}\mathscr{G}(x-G'(x),k)$. Then

$$\|G(x)-G'(x)\| \le g_m\|x-G'(x)\| \le g_m g_m^c\|x\|.$$

$\square$

Lemma 4.0.2. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. For each $m\in\mathbb{N}$ we have

$$\max\{\tfrac{1}{2\kappa}\tilde{\mu}_m, \tilde{\mu}_m^d, \mu_m\} \le \nu_m \le g_m^c + g_m\tilde{\mu}_m^d, \tag{4.4}$$

where $\kappa = 1$ or 2, if $\mathbb{F}$ is real or complex, respectively.

Proof. For the lower bounds, $\tilde{\mu}_m^d \le \nu_m$ follows taking $x=0$. For $\mu_m\le\nu_m$, writing $\mathbf{1}_A=\mathbf{1}_{A\setminus B}+\mathbf{1}_{A\cap B}$ and $\mathbf{1}_B=\mathbf{1}_{B\setminus A}+\mathbf{1}_{A\cap B}$, taking $x=\mathbf{1}_{A\cap B}$ in ν_m we obtain the bound. To show that $\nu_m\ge\frac{1}{2\kappa}\tilde{\mu}_m$, we can repeat the argument of Proposition 2.2.5: take $|A|\le|B|\le m$ and $\varepsilon\in\Psi_A$, $\varepsilon'\in\Psi_B$. Using Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| = \left\|\sum_{n\in A}\frac{\varepsilon_n}{\varepsilon'_n}\varepsilon'_n\mathbf{e}_n\right\| \le 2\kappa\sup_{D\subseteq A}\|\mathbf{1}_{\varepsilon' D}\|.$$

Now, since $\mathbf{1}_{\varepsilon' D}=\mathbf{1}_{\varepsilon' D\setminus B}+\mathbf{1}_{\varepsilon'(B\cap D)}$ and $|D\setminus B|\le|B\setminus D|\le m$,

$$\|\mathbf{1}_{\varepsilon' D}\| = \|\mathbf{1}_{\varepsilon'(D\setminus B)}+\mathbf{1}_{\varepsilon'(D\cap B)}\| \le \nu_m\|\mathbf{1}_{\varepsilon'(B\setminus D)}+\mathbf{1}_{\varepsilon'(D\cap B)}\| = \|\mathbf{1}_{\varepsilon' B}\|.$$

Hence, $\tilde{\mu}_m\le 2\kappa\nu_m$.

Finally, for the upper bound, for each $|A|\le|B|\le m$, $\varepsilon\in\Psi_A, \eta\in\Psi_B$, $|x|_\infty\le 1$, $A\cup B\cup x$ we have $\|x\|\le g_m^c\|\mathbf{1}_{\eta B}+x\|$ and $\|\mathbf{1}_{\varepsilon A}\|\le\tilde{\mu}_m^d\|\mathbf{1}_{\eta B}\|\le\tilde{\mu}_m^d g_m\|\mathbf{1}_{\eta B}+x\|$. Hence the inequality $\nu_m\le g_m^c+g_m\tilde{\mu}_m^d$ is easily obtained. $\square$

Remark 4.0.3. We do not know whether $\nu_m\ge\tilde{\mu}_m$ (or even $\mathbf{L}_m\ge\tilde{\mu}_m$) may hold in general.

The next lemma is a straightforward extension of the equality defining ν_m.

Lemma 4.0.4. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. Let $x\in\mathbb{X}$ and $\alpha\ge\max_n|\mathbf{e}_n^*(x)|$. Then

$$\big\|x+z\big\| \le \nu_m\big\|x+\alpha\mathbf{1}_{\eta B}\big\|, \quad \forall\,\eta\in\Psi_B,$$

and for all B and z such that $|\operatorname{supp}(z)|\le|B|\le m$, $B\cup x\cup z$ and $|z|_\infty\le\alpha$.

Proof. Take x, z, α and η as in the statement. Applying Corollary 1.3.2,

$$\left\|\frac{x}{\alpha}+\frac{z}{\alpha}\right\| \le \sup_{\varepsilon\in\Psi_A}\left\|\frac{x}{\alpha}+\mathbf{1}_{\varepsilon A}\right\| \le \nu_m\left\|\frac{x}{\alpha}+\mathbf{1}_{\eta B}\right\|.$$

This completes the proof. $\square$

In a similar fashion one shows the following.

Lemma 4.0.5. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. Let $z\in\mathbb{X}$ and $B\subset\mathbb{N}$ such that $|\operatorname{supp}(z)|\le|B|\le m$. Then

$$\big\|z\big\| \le \tilde{\mu}_m\max|\mathbf{e}_n^*(z)|\,\big\|\mathbf{1}_{\eta B}\big\|, \quad \forall\,\eta\in\Psi_B.$$

To give a relation between $\tilde{\mu}_m$ and μ_m, we need to consider the unconditionality for constant coefficients parameter:

$$\gamma_m = \sup\left\{\frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|} : B \subseteq A,\ |A| \leq m,\ \varepsilon \in \Psi_A\right\}, \tag{4.5}$$

and observe that $\gamma_m \leq \hat{g}_m$. We also have the following lemma.

Lemma 4.0.6. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis of a Banach space $\mathbb{X}$. Let $\kappa = 1$ or 2, if $\mathbb{F}$ is real or complex, respectively. Then,

$$\left\|\mathbf{1}_{\varepsilon B}\right\| \leq 2\kappa\,\gamma_m \left\|\mathbf{1}_{\eta A}\right\|, \quad \forall\, B \subseteq A,\ |A| \leq m,\ \varepsilon \in \Psi_B, \eta \in \Psi_A. \tag{4.6}$$

Proof. Using Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon B}\| \leq 2\kappa \sup_{D \subseteq A} \|\mathbf{1}_{\eta D}\| \leq 2\kappa\gamma_m \|\mathbf{1}_{\eta A}\|.$$

Hence, we obtain (4.6). □

Lemma 4.0.7. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis of a Banach space $\mathbb{X}$. If κ be as in Lemma 4.0.6, then,

$$\tilde{\mu}_m \leq 4\,\kappa^2\,\gamma_m\,\mu_m, \quad \forall\, m = 1, 2, \ldots \tag{4.7}$$

Proof. Take $A, B \subset \mathbb{N}$ with $|A| \leq |B| \leq m$ and $\varepsilon \in \Psi_A, \eta \in \Psi_B$. Using Corollary 1.3.3 and μ_m,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D \subseteq A} \|\mathbf{1}_D\| \leq 2\kappa\mu_m \|\mathbf{1}_B\|. \tag{4.8}$$

Now, using Lemma 4.0.6,

$$\|\mathbf{1}_B\| \leq 2\kappa\gamma_m \|\mathbf{1}_{\eta B}\|. \tag{4.9}$$

Using (4.9) in (4.8), we obtain the result. □

Finally, we present a collection of implications that involves the truncation operator (see (2.7)). If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is an SM-basis in a Banach space $\mathbb{X}$, for each number α and $x \in \mathbb{X}$,

$$T_\alpha(x) := \alpha \mathbf{1}_{\varepsilon\Lambda_\alpha} + (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_\alpha})(x),$$

where $\Lambda_\alpha = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$ and $\varepsilon \equiv \{\mathrm{sign}(\mathbf{e}_j^*(x))\}$.

Lemma 4.0.8. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis of a Banach space $\mathbb{X}$. For all $\alpha > 0$ and $x \in \mathbb{X}$ we have

$$\|T_\alpha(x)\| \leq g^c_{|\Lambda_\alpha|}\,\|x\|, \quad \|(\mathrm{Id}_{\mathbb{X}} - T_\alpha)(x)\| \leq g_{|\Lambda_\alpha|}\,\|x\|, \tag{4.10}$$

where $\Lambda_\alpha = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$. Moreover, for every finite set A,

$$\|T_\alpha(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\| \leq k^c_{|A \cup \Lambda_\alpha|}\,\|x\|, \tag{4.11}$$

and if $x \in \mathbb{X}$ and $\varepsilon \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$, then

$$\min_{n\in\Lambda} |\mathbf{e}_n^*(x)| \left\|\mathbf{1}_{\varepsilon\Lambda}\right\| \le \tilde{g}_m \|x\|, \quad \forall\, \Lambda \in \mathscr{G}(x,m). \tag{4.12}$$

Proof. The proof is exactly the same as in Lemmas 2.1.13 and 2.1.14 of Chapter 2, but we rewrite the proof showing the dependence on $|\Lambda_\alpha|$.

Take $x \in \mathbb{X}$ and Λ a greedy set of x. Set $\alpha \in \mathbb{F}$. Then, we have

$$T_\alpha(x) = \int_0^1 \Big[\sum_n \chi_{[0,\frac{\alpha}{|\mathbf{e}_n^*(x)|}]}(s)\, \mathbf{e}_n^*(x)\mathbf{e}_n\Big]\, ds = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_{\alpha,s}})(x)\, ds, \tag{4.13}$$

where $\Lambda_{\alpha,s} = \{n : |\mathbf{e}_n^*(x)| > \frac{\alpha}{s}\}$ for each $s \in (0,1]$. Of course, $\Lambda_{\alpha,s} \in \mathscr{G}(x,k_s)$ with $|k_s| = |\Lambda_{\alpha,s}|$ and $\Lambda_{\alpha,s} \subseteq \Lambda_\alpha \subset \Lambda$. The result follows from Minkowski's integral inequality applied to (4.13) and using the following two formulas derived from it:

$$(\mathrm{Id}_{\mathbb{X}} - T_\alpha)(x) = \int_0^1 P_{\Lambda_{\alpha,s}}(x)\, ds,$$

and

$$T_\alpha(\mathrm{Id}_{\mathbb{X}} - P_A)(x) = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{\Lambda_{\alpha,s}})(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\, ds = \int_0^1 (\mathrm{Id}_{\mathbb{X}} - P_{A\cup\Lambda_{\alpha,s}})(x)\, ds.$$

To show (4.12), we proceed as follows: if $\alpha = \min_{j\in\Lambda} |\mathbf{e}_j^*(x)|$ with $\Lambda \in \mathscr{G}(x,m)$,

$$\alpha \mathbf{1}_{\varepsilon\Lambda} = T_\alpha(x) - P_{\Lambda^c}(x) = \int_0^1 (P_\Lambda(x) - P_{\Lambda_{\alpha,s}}(x))ds.$$

Since $\|P_\Lambda(x) - P_{\Lambda_{\alpha,s}}(x)\| \le \tilde{g}_m(x)$, the results follows.

□

Remark 4.0.9. Observe that, from (4.11), one has the trivial estimate

$$\|T_\alpha(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\| \le g^c_{|\Lambda_\alpha|}\, k^c_{|A|}\, \|x\|. \tag{4.14}$$

Being multiplicative, (4.14) is typically worse than (4.11) (if say both k_m^c and g_m^c grow fast as $m \to \infty$). However in some cases it may better, as for example when $g^c_{|\Lambda_\alpha|} = 1$.

4.1 Main results about $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$

Here, we present some upper and lower bounds for $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$ and we study the optimality of these estimates.

4.1.1 First result and its optimality

Theorem 4.1.1. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. For all $m \ge 1$ we have

$$\mathbf{L}_m \le 1 + 3\mathfrak{K}\, m \quad \text{and} \quad \widetilde{\mathbf{L}}_m \le 1 + 2\mathfrak{K}\, m, \tag{4.15}$$

where $\mathfrak{K} = \sup_{k,n} \|\mathbf{e}_k\| \|\mathbf{e}_n^*\|_*$. Moreover, there exists $(\mathbb{X}, \mathscr{B})$ for which both equalities are attained.

Proof. The first estimate in (4.15) is implicit in the first papers in the topic ([76, 77] or [67, (1.8)]). We sketch below the elementary proof, as it also gives the second estimate.

Take $x \in \mathbb{X}$ and $\Gamma \in \mathscr{G}(x,m)$. Pick any $z = \sum_{n\in A} z_n \mathbf{e}_n$ such that $|A| \leq m$. Of course, we have the following known decomposition:

$$x - P_\Gamma(x) = P_{(A\cup\Gamma)^c}(x-z) + P_{A\setminus\Gamma}(x). \tag{4.16}$$

On the one hand, it is clear that $\|P_{(A\cup\Gamma)^c}(x-z)\| \leq k_{2m}^c \|x-z\|$. On the other hand,

$$\begin{aligned} \left\|P_{A\setminus\Gamma}(x)\right\| &\leq \sum_{k\in A\setminus\Gamma} |\mathbf{e}_k^*(x)| \|\mathbf{e}_k\| \leq \sup_k \|\mathbf{e}_k\| \sum_{n\in\Gamma\setminus A} |\mathbf{e}_n^*(x)| \\ &\leq \sup_{k,n} \|\mathbf{e}_k\| \|\mathbf{e}_n^*\|_* \, m \, \|x-z\|, \end{aligned} \tag{4.17}$$

since $\mathbf{e}_n^*(x) = \mathbf{e}_n^*(x-z)$ when $n \notin A$. On the other hand, if we define $k_1 = \sup_{n\geq 1} \|\mathbf{e}_n\| \|\mathbf{e}_n^*\|_* \leq \mathfrak{K}$, since $k_m \leq k_1 m$, we obtain that

$$g_m^c \leq k_m^c \leq 1 + k_1 m \leq 1 + \mathfrak{K}m. \tag{4.18}$$

Using now (4.18), we obtain that

$$\|x - P_\Gamma(x)\| \leq k_{2m}^c + \mathfrak{K}m \leq 1 + 2\mathfrak{K}m + \mathfrak{K}m = 1 + 3\mathfrak{K}m.$$

Thus, $\mathbf{L}_m \leq 1 + 3\mathfrak{K}m$.

To show the bound of $\widetilde{\mathbf{L}}_m$, taking any set A with $|A| \leq m$, we have a similiar decomposition than in 4.16:

$$x - P_\Gamma(x) = P_{(A\cup\Gamma)^c}(x - P_A(x)) + P_{A\setminus\Gamma}(x). \tag{4.19}$$

We only have to estimate the quantity $P_{(A\cup\Gamma)^c}(x - P_A(x))$. For that,

$$\|P_{(A\cup\Gamma)^c}(x - P_A(x))\| = \|(\mathrm{Id}_{\mathbb{X}} - P_{\Gamma\setminus A})(x - P_A(x))\| \leq g_m^c \|x - P_A(x)\|. \tag{4.20}$$

Using (4.18), (4.17) and (4.20) in 4.19, we conclude that $\widetilde{\mathbf{L}}_m \leq 1 + 2\mathfrak{K}m$.

□

We present an example of a basis in a Banach space showing the optimality of Theorem 4.1.1.

Example 4.1.2 (The summing basis)**.** Let $\mathbb{X}$ be the closure of the set of all finite sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in c_{00}$ with

$$\|\mathbf{a}\| := \sup_{M\geq 1} \Big| \sum_{n=1}^{M} a_n \Big| < \infty. \tag{4.21}$$

The standard canonical system $\{\mathbf{e}_n,\mathbf{e}_n^*\}$ satisfies $\|\mathbf{e}_m\|\equiv 1$, $\|\mathbf{e}_1^*\|_*=1$ and $\|\mathbf{e}_n^*\|_*=2$ if $n\geq 2$ (so $\mathfrak{K}=2$, with the notation in Theorem 4.1.1). The terminology comes from the fact that $\mathbb{X}$ is isometrically isomorphic[2] to the span of the "summing system" $\{\mathbf{s}_n:=\sum_{k\geq n}\mathbf{e}_k\}_{n=1}^{\infty}$ in ℓ_∞ (see [63, p. 20]).

For this example we have:

a) $\mu_m=1$ and $\tilde{\mu}_m=m$.

b) $g_m=\tilde{g}_m=k_m=2m$ and $g_m^c=k_m^c=1+2m$.

c) $\nu_m=\widetilde{\mathbf{L}}_m=1+4m$ and $\mathbf{L}_m=1+6m$.

So, equalities hold everywhere in Theorem 4.1.1.

a) It is clear that $\|\mathbf{1}_A\|=|A|$, so the basis is democratic and $\mu_m=1$ for all $m\in\mathbb{N}$. Now, we have that

$$\tilde{\mu}_m\geq\frac{\|\sum_{n=1}^{m}\mathbf{e}_n\|}{\|\sum_{n=1}^{m}(-1)^n\mathbf{e}_n\|}=\frac{m}{1}=m.$$

Now, since for any $|A|\leq|B|\leq m$ and $\varepsilon\in\Psi_A,\varepsilon'\in\Psi_B$ we have that $\sup_{\varepsilon\in\Psi_A,|A|\leq m}\|\mathbf{1}_{\varepsilon A}\|\leq m$ and $\inf_{\varepsilon'\in\Psi_B,|B|\geq m}\|\mathbf{1}_{\varepsilon' B}\|\geq 1$, then we conclude that $\tilde{\mu}_m=m$.

b) We know from Lemma 4.0.1 and (4.18), that $g_m\leq\tilde{g}_m\leq k_m\leq 2m$. To see the lower bound, pick the vector $\mathbf{a}=(1,-2,2,\dots,-2,2,0,\dots)$, where $|\{a_n=\pm 2\}|=2m$ and $\|\mathbf{a}\|=1$. Then, if we take $\Gamma=\{n:a_n=2\}\in\mathscr{G}(\mathbf{a},m)$,

$$g_m\geq\|P_\Gamma(\mathbf{a})\|=\|(0,2,0,\dots,2,0,0\dots)\|=2m.$$

Similarly, $g_m^c\leq k_m^c\leq 1+2m$ by (4.18), and setting $\Gamma'=\{n:a_n=-2\}\in\mathscr{G}(\mathbf{a},m)$ we conclude

$$g_m^c\geq\|(I-P_{\Gamma'})(\mathbf{a})\|=\|(1,2,0,\dots,2,0,0\dots)\|=1+2m.$$

c) Next we have $\widetilde{\mathbf{L}}_m\leq 1+4m$ by Theorem 4.1.1. For the lower bound we pick

$$x=\big(\overbrace{\tfrac{1}{2},0,\tfrac{1}{2}}\,;\,\dots\,;\,\overbrace{\tfrac{1}{2},0,\tfrac{1}{2}}\,;\,\tfrac{1}{2},0,0,\dots\big)\quad\text{and}\quad\mathbf{1}_B=\big(\overbrace{0,1,0}\,;\,\dots\,;\,\overbrace{0,1,0}\,;\,0,\dots\big),$$

where we have m times $\left(\frac{1}{2},0,\frac{1}{2}\right)$ in x and m times $(0,1,0)$ in $\mathbf{1}_B$. Then, $\|x-\mathbf{1}_B\|=1/2$, while $\|x+\mathbf{1}_A\|=\frac{1}{2}+2m$ for any $|A|=m$ with $A\mathbin{\dot\cup} B\mathbin{\dot\cup}\operatorname{supp}(x)$. So, if we define $y:=x-\mathbf{1}_B+\mathbf{1}_A$, $B\in\mathscr{G}(y,m)$ and

$$\|x+\mathbf{1}_A\|\leq\nu_m\|x-\mathbf{1}_B\|.$$

Thus,

$$\nu_m\geq\frac{\|x+\mathbf{1}_A\|}{\|x-\mathbf{1}_B\|}=1+4m.$$

Also, since $\widetilde{\mathbf{L}}_m\geq\nu_m\geq 1+4m$ and $\kappa=2$, using Theorem 4.1.1, $\widetilde{\mathbf{L}}_m=\nu_m=1+4m$.

Finally, $\mathbf{L}_m\leq 1+6m$ by Theorem 4.1.1. To show the equality, let

$$x=\big(\overbrace{\tfrac{1}{2},1,\tfrac{1}{2}}\,;\,\dots\,;\,\overbrace{\tfrac{1}{2},1,\tfrac{1}{2}}\,;\,\tfrac{1}{2};\,\overbrace{-1,1},\,\dots\,,\overbrace{-1,1},\,0,0,\dots\big),$$

[2] Via the map $\mathbf{a}\in\mathbb{X}\mapsto T\mathbf{a}=(\sum_{i=1}^{n}a_i)_{n\in\mathbb{N}}\in\ell_\infty$, since $T\mathbf{e}_n=\mathbf{s}_n$.

where we have m times $\left(\frac{1}{2},1,\frac{1}{2}\right)$ and m times $(-1,1)$. Pick $\Gamma=\{n : x_n=-1\}\in\mathscr{G}(x,m)$. Then

$$\|x-P_\Gamma(x)\| \,=\, 3m+\tfrac{1}{2},$$

while

$$\sigma_m(x)\leq \big\|x-2\big(\overbrace{0,1,0}\,;\,\dots\,;\,\overbrace{0,1,0}\,;\,0,0,\dots\big)\big\| \,=\, \frac{1}{2}.$$

Thus, $\mathbf{L}_m\geq \|x-P_\Gamma(x)\|/\sigma_m(x)\,\geq\, 6m+1$.

4.1.2 Second and third results and their optimality

We present two results: both of them are given by a multiplicative upper bound, but in the first one we will use the parameter of the symmetry for largest coefficients, and in the other one we use the super-democratic parameter. Also, we will discuss with two examples the optimality of them.

Theorem 4.1.3. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. For all $m\geq 1$ we have

$$\mathbf{L}_m\leq k_{2m}^c\,\nu_m,\quad \text{and}\quad \widetilde{\mathbf{L}}_m\leq g_m^c\,\nu_m. \tag{4.22}$$

Moreover, there exists $(\mathbb{X},\mathscr{B})$ for which both equalities are attained.

Proof. Let $x\in\mathbb{X}$ and $\Gamma\in\mathscr{G}(x,m)$, and call $\alpha=\min_{n\in\Gamma}|\mathbf{e}_n^*(x)|$. Pick any $z\in\mathbb{X}$ with $\operatorname{supp}(z)=A$ and $|A|\leq|\Gamma|=m$. Then we can write

$$x-P_\Gamma(x)\,=\,(\mathrm{Id}_\mathbb{X}-P_{A\cup\Gamma})(x)\,+\,P_{A\setminus\Gamma}(x). \tag{4.23}$$

Since $\{|(\mathrm{Id}_\mathbb{X}-P_{A\cup\Gamma})(x)|_\infty, |P_{A\setminus\Gamma}(x)|_\infty\}\leq\alpha$ and $|A\setminus\Gamma|\leq|\Gamma\setminus A|$, we can apply Lemma 4.0.4 with $\eta\equiv\{\operatorname{sign}(\mathbf{e}_n^*(x))\}$ to obtain

$$\begin{aligned}\|x-P_\Gamma(x)\| \;&\leq\; \nu_m\,\big\|\alpha\mathbf{1}_{\eta(\Gamma\setminus A)}+P_{(A\cup\Gamma)^c}(x)\big\|\\ &=\; \nu_m\,\big\|T_\alpha\big[(\mathrm{Id}_\mathbb{X}-P_A)(x)\big]\big\|\,=\,\nu_m\,\big\|T_\alpha\big[(\mathrm{Id}_\mathbb{X}-P_A)(x-z)\big]\big\|\\ &\leq\; \nu_m\,k_{|A\cup\Gamma|}^c\,\|x-z\|\,\leq\,\nu_m\,k_{2m}^c\,\|x-z\|,\end{aligned} \tag{4.24}$$

using 4.11 in the second inequality. Thus, taking the infimum over all z we conclude that

$$\mathbf{L}_m\leq\nu_m\,k_{2m}^c.$$

The estimate for $\widetilde{\mathbf{L}}_m$ follows with the same argument than before: for any set A with $|A|\leq|\Gamma|=m$ we have

$$\|x-P_\Gamma(x)\|\leq\nu_m\,\big\|T_\alpha\big[(\mathrm{Id}_\mathbb{X}-P_A)(x)\big]\big\|\,\leq\,\nu_m\,g_m^c\,\|x-P_A(x)\|,$$

using (4.10). This implies that $\widetilde{\mathbf{L}}_m\leq\nu_m g_m^c$, and establishes the theorem. □

Remark 4.1.4. Notice that we could use in (4.24) the estimate in Remark 4.0.9, leading to the slightly smaller bound

$$\mathbf{L}_m\,\leq\,\min\{k_{2m}^c,k_m^c g_m^c\}\,\nu_m.$$

For instance, if we assume $g_m^c = 1$ for some m (or equivalently, for all m), since $k_m^c \le k_{2m}^c$, we obtain

$$\mathbf{L}_m \le k_m^c \, \nu_m,$$

which we shall use in Corollary 4.1.9.

Theorem 4.1.5. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis of a Banach space $\mathbb{X}$. For all $m \ge 1$ we have

$$\mathbf{L}_m \le k_{2m}^c + \tilde{g}_m \, \tilde{\mu}_m, \quad \widetilde{\mathbf{L}}_m \le g_m^c + \tilde{g}_m \, \tilde{\mu}_m, \tag{4.25}$$

and

$$\max\{k_m^c, \, \widetilde{\mathbf{L}}_m\} \le \mathbf{L}_m, \quad \max\{g_m^c, \, \nu_m\} \le \widetilde{\mathbf{L}}_m. \tag{4.26}$$

Proof. Take $x \in \mathbb{X}$, $\Gamma \in \mathscr{G}(x,m)$ and $z = \sum_{n\in A} z_n \mathbf{e}_n$ with $|A| \le m$. With the same decomposition as in (4.23), it is clear that

$$\|(\mathrm{Id}_{\mathbb{X}} - P_{A\cup\Gamma})(x)\| = \|(\mathrm{Id}_{\mathbb{X}} - P_{A\cup\Gamma})(x-z)\| \le k_{2m}^c \, \|x-z\|. \tag{4.27}$$

Hence, we only need to estimate the quantity $\|P_{A\setminus\Gamma}(x)\|$. We pick any set $\tilde{\Gamma} \in \mathscr{G}(x-z, |A\setminus\Gamma|)$, and use the following elementary observation (see [44, p. 453]):

$$\max_{n\in A\setminus\Gamma} |\mathbf{e}_n^*(x)| \le \min_{n\in\tilde{\Gamma}} |\mathbf{e}_n^*(x-z)|. \tag{4.28}$$

Then, using Corollary 1.3.3 and the parameter $\tilde{\mu}_m$ with $\eta \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x-z))\}$,

$$\|P_{A\setminus\Gamma}(x)\| \le \max_{n\in A\setminus\Gamma} |\mathbf{e}_n^*(x)| \sup_{\varepsilon\in\Psi_{A\setminus\Gamma}} \|\mathbf{1}_{\varepsilon(A\setminus\Gamma)}\| \le \tilde{\mu}_m \min_{n\in\tilde{\Gamma}} |\mathbf{e}_n^*(x-z)| \|\mathbf{1}_{\eta\tilde{\Gamma}}\|. \tag{4.29}$$

By Lemma 4.0.8,

$$\min_{n\in\tilde{\Gamma}} |\mathbf{e}_n^*(x-z)| \|\mathbf{1}_{\eta\tilde{\Gamma}}\| \le \tilde{g}_m \|x-z\|. \tag{4.30}$$

So, using (4.30) in (4.29),

$$\|P_{A\setminus\Gamma}(x)\| \le \tilde{\mu}_m \tilde{g}_m \|x-z\|. \tag{4.31}$$

Adding up (4.27) and (4.31) and taking the infimum over all $z \in \Sigma_m$, we obtain

$$\mathbf{L}_m \le k_{2m}^c + \tilde{\mu}_m \, \tilde{g}_m,$$

as asserted in (4.25).

The estimate for $\widetilde{\mathbf{L}}_m$ is again similar: given a set A with $|A| \le |\Gamma| = m$, we can replace (4.27) by

$$\|(\mathrm{Id}_{\mathbb{X}} - P_{A\cup\Gamma})(x)\| = \|(\mathrm{Id}_{\mathbb{X}} - P_{\Gamma\setminus A})(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\| \le g_m^c \, \|x - P_A(x)\|, \tag{4.32}$$

since $\Gamma \setminus A \in \mathscr{G}\big(x - P_A(x)\big)$. The second estimate in (4.30) is valid in this case setting $z = P_A(x)$ and $\tilde{\Gamma} = \Gamma \setminus A$. Thus we conclude $\widetilde{\mathbf{L}}_m \le g_m^c + \tilde{\mu}_m \, \tilde{g}_m$.

The lower bounds are quite elementary, and most of them have appeared before in the literature. We sketch the proof of those we did not find explicitly in this generality.

- $\mathbf{L}_m \geq k_m^c$. This bound can be proved using the same arguments than in Theorem 2.3.7. Indeed, take $x \in \mathbb{X}$ with finite support and $|A| \leq m$. Consider also a set C such that $C \cap \operatorname{supp}(x) = \emptyset$ and $|A \cup C| = m$. Define the element

$$y := (\mathrm{Id}_{\mathbb{X}} - P_A)(x) + \sum_{n \in A \cup C} (\mathbf{e}_n^*(x) + \alpha)\mathbf{e}_n,$$

where

$$\alpha > \sup_{j \in A} |\mathbf{e}_j^*(x)| + \sup_{j \in A^c} |\mathbf{e}_j^*(x)|.$$

Hence, $A \cup C \in \mathscr{G}(y,m)$ and

$$\|x - P_A(x)\| \leq \mathbf{L}_m \sigma_m(y) \leq \mathbf{L}_m \|y - \alpha \mathbf{1}_A - \sum_{n \in C} (\mathbf{e}_n^*(x) + \alpha)\mathbf{e}_n\| = \mathbf{L}_m \|x\|.$$

Using density, the proof is over.

- $\widetilde{\mathbf{L}}_m \geq \nu_m$. Let $|A| \leq |B| \leq m$, $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$ and $x \in \mathbb{X}$ such that A, B and $\operatorname{supp}(x)$ are pairwise disjoint and $|x|_\infty \leq 1$. We must show that

$$\|\mathbf{1}_{\varepsilon A} + x\| \leq \widetilde{\mathbf{L}}_m \|\mathbf{1}_{\eta B} + x\|. \tag{4.33}$$

For every $j \geq 1$ we can find a set C_j with $|C_j| = m - |B|$, disjoint with $A \cup B$, and such that $\max_{n \in C_j} |\mathbf{e}_n^*(x)| \leq 1/j$. We set

$$y_j = \mathbf{1}_{\varepsilon A} + \mathbf{1}_{\varepsilon' B} + (\mathrm{Id}_{\mathbb{X}} - P_{C_j})(x) + \mathbf{1}_{C_j},$$

and select $G_m \in \mathscr{G}_m$ such that $G_m(y_j) = \mathbf{1}_{\varepsilon' B} + \mathbf{1}_{C_j}$. Then

$$\begin{aligned} \|\mathbf{1}_{\varepsilon A} + (I - P_{C_j})(x)\| &= \|y_j - G_m(y_j)\| \\ &\leq \widetilde{\mathbf{L}}_m \widetilde{\sigma}_m(y_j) \\ &\leq \widetilde{\mathbf{L}}_m \|\mathbf{1}_{\varepsilon' B} + (\mathrm{Id}_{\mathbb{X}} - P_{C_j})(x)\|. \end{aligned}$$

Since $\lim_{j \to \infty} P_{C_j}(x) = 0$ we obtain (4.33).

- $\widetilde{\mathbf{L}}_m \geq g_m^c$. We must show that for every $x \in \mathbb{X}$ and every $\Gamma \in \mathscr{G}(x,k)$ with $k \leq m$, we have

$$\|x - P_\Gamma(x)\| \leq \widetilde{\mathbf{L}}_m \|x\|. \tag{4.34}$$

Let $\alpha = \min_{n \in \Gamma} |\mathbf{e}_n^*(x)|$. Notice that for every $j \geq 1$ we can find a set $C_j \subset \Gamma^c$, with $|C_j| = m - k$, and $\max_{n \in C_j} |\mathbf{e}_n^*(x)| \leq \alpha / j$. Let

$$y_j = x - P_{C_j}(x) + \alpha \mathbf{1}_{C_j},$$

so that $\Gamma \cup C_j \in \mathscr{G}(y_j, m)$. Thus

$$\begin{aligned} \|y_j - P_{\Gamma\cup C_j}(y_j)\| &\leq \widetilde{\mathbf{L}}_m \widetilde{\sigma}_m(y_j) \\ &\leq \widetilde{\mathbf{L}}_m \|y_j - P_{C_j}(y_j)\|, \end{aligned}$$

which is the same as

$$\|x - P_\Gamma(x) - P_{C_j}(x)\| \leq \widetilde{\mathbf{L}}_m \|x - P_{C_j}(x)\|.$$

Since $\lim_{j\to\infty} P_{C_j} x = 0$ (in $\mathbb{X}$) we obtain (4.34).

□

Corollary 4.1.6. If $\mathscr{B}$ is an SM-basis and C_q-quasi-greedy in a Banach space $\mathbb{X}$, then

$$\max\{k_m^c, \mu_m\} \leq \mathbf{L}_m \leq k_{2m}^c + 8\,\kappa^2 C_q^2\, \mu_m \tag{4.35}$$

and

$$\max\{g_m^c, \mu_m\} \leq \widetilde{\mathbf{L}}_m \leq g_m^c + 8\,\kappa^2 C_q^2\, \mu_m, \tag{4.36}$$

where $\kappa = 1$ if $\mathbb{F} = \mathbb{R}$, and $\kappa = 2$ if $\mathbb{F} = \mathbb{C}$.

Proof. Using Theorem 4.1.5 and Lemma 4.0.2, we obtain the lower bounds of $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$. For the upper bounds, first, using Lemma 4.0.1, $\tilde{g}_m \leq 2C_q$. On the other hand, by Lemma 4.0.6, $\tilde{\mu}_m \leq 4\kappa^2 C_q \mu_m$. Hence, with both estimates, we obtain the result. □

Remark 4.1.7. This corollary recovers the classical results for quasi-greedy bases (see [44], [35]) and, also, the result shows an improvement with respect to the upper bound.

The optimality that we have commented in the Theorem 4.1.3 is based on the following corollaries and in the Example 4.1.11.

Corollary 4.1.8. If for some m_0 we have $k_{m_0}^c = 1$, then $k_m^c \equiv 1$ and

$$\mathbf{L}_m = \widetilde{\mathbf{L}}_m = \nu_m,\ \forall m \geq 1.$$

Proof. Since $1 \leq k_1^c \leq k_{m_0}^c = 1$, it follows that $k_1^c = 1$. Using $\mathrm{Id}_{\mathbb{X}} - P_A = \prod_{n\in A}(\mathrm{Id}_{\mathbb{X}} - P_n)$, it follows that $k_m^c \leq (k_1^c)^m = 1$. Since $k_m \geq 1$ for all $m = 1, 2, \ldots$, $k_m^c \equiv 1$. Now, by (4.26), $\nu_m \leq \widetilde{\mathbf{L}}_m \leq \mathbf{L}_m$. Since $k_m^c = 1$ for all $m \in \mathbb{N}$, Theorem 4.1.3 implies that $\mathbf{L}_m \leq \nu_m$. This finishes the proof. □

Corollary 4.1.9. If for some m_0 we have $\nu_{m_0} = 1$, then $\nu_m \equiv 1$ and

$$\mathbf{L}_m = k_m^c, \quad \text{and} \quad \widetilde{\mathbf{L}}_m = g_m^c = 1,\ \forall m \geq 1.$$

Proof. Since $\nu_1 \leq \nu_{m_0} = 1$ it follows that $\nu_1 = 1$. A simple induction argument (see [4, Lemma 2.1]), shows that

$$\nu_m \leq (\nu_1)^m,$$

so we shall have $\nu_m \equiv 1$. From here, arguing as in [4, Theorem 2.3] one obtains $\widetilde{\mathbf{L}}_m \equiv 1$, and by (4.26) also $g_m^c \equiv 1$. Thus, we can invoke Remark 4.1.4 to obtain that $\mathbf{L}_m \le k_m^c$. This together with the lower bound $\mathbf{L}_m \ge k_m^c$ in (4.26) establishes the corollary. □

Corollary 4.1.10. If $\mathscr{B}$ is an SM-super-democratic basis in a Banach space $\mathbb{X}$, then

$$\mathbf{L}_m \approx k_m.$$

Proof. Since $\tilde{\mu}_m = O(1)$, using Theorem 4.1.5 and the fact that $\tilde{g}_m \le k_m$, we obtain the result. □

Example 4.1.11 (Canonical basis in $\ell_1 \oplus c_0$)**.** Consider the space formed by pairs of sequences $(x,y) \in \ell_1 \times c_0$, endowed with the norm $\|(x,y)\| = \|x\|_1 + \|y\|_\infty$. Write the canonical basis as $\mathscr{B} = \{(\mathbf{e}_m,0),(0,\mathbf{f}_n)\}_{m,n=1}^\infty$, where $(\mathbf{e}_m)_{m=1}^\infty$ and $(\mathbf{f}_n)_{n=1}^\infty$ are the canonical bases of the spaces ℓ_1 and c_0, respectively.

The canonical basis in $\ell_1 \oplus c_0$ satisfies:

a) $\mu_m = \tilde{\mu}_m = m$,

b) $g_m = \tilde{g}_m = k_m = g_m^c = k_m^c = 1$,

c) $\nu_m = \widetilde{\mathbf{L}}_m = \mathbf{L}_m = 1 + \tilde{\mu}_m = 1 + m$.

So, equalities hold everywhere in Theorem 4.1.5.

The item b) is clear, since the canonical basis is 1-unconditional. For the item a) just notice that

$$1 \le \|\mathbf{1}_A\| = \|\mathbf{1}_{\varepsilon A}\| \le |A|,$$

with the lower bound attained when $\mathbf{1}_A \in c_0$, and the upper bound when $\mathbf{1}_A \in \ell_1$. Finally, since the basis is 1-unconditional, by Theorem 4.1.5,

$$\widetilde{\mathbf{L}}_m \le \mathbf{L}_m \le 1 + \tilde{\mu}_m \le 1 + m.$$

On the other hand, by Corollary 4.1.8,

$$\widetilde{\mathbf{L}}_m = \mathbf{L}_m = \nu_m,$$

hence, we only need to show that $\nu_m \ge m+1$. Let $\mathbf{1}_A = \sum_{n=1}^m (\mathbf{e}_n,0)$, $\mathbf{1}_B = \sum_{n=1}^m (0,\mathbf{f}_n)$, and $x = (0,\mathbf{f}_{m+1})$, then

$$\nu_m \ge \frac{\|\mathbf{1}_A + x\|}{\|\mathbf{1}_B + x\|} = m+1.$$

Example 4.1.12 (Canonical basis in $\ell_1 \oplus \ell_q$, $1 \le q < \infty$)**.** This variant of the previous example also admits explicit Lebesgue parameters, but equality fails in (4.25).

As before, we write the canonical basis in $\ell_1 \oplus \ell_q$ as $\mathscr{B} = \{(\mathbf{e}_m,0),(0,\mathbf{f}_n)\}_{m,n=1}^\infty$, with $1 \le q < \infty$ and $\|(x,y)\| = \|x\|_1 + \|y\|_q$ for $(x,y) \in \ell_1 \times \ell_q$. Then, $\mathscr{B}$ satisfies

a) $\mu_m = \tilde{\mu}_m = m^{1/q'}$,

b) $g_m = \tilde{g}_m = k_m = g_m^c = k_m^c = 1$,

c) $\nu_m = \widetilde{\mathbf{L}}_m = \mathbf{L}_m = (m+1)^{1/q'}$.

So equality holds in Theorem 4.1.3, but not in Theorem 4.1.5.

We only prove the item c) since the other two are easy. By Corollary 4.1.8, we only need to estimate ν_m. From below, we choose as before $\mathbf{1}_A = \sum_{n=1}^{m}(\mathbf{e}_n, 0)$, $\mathbf{1}_B = \sum_{n=2}^{m+1}(0, \mathbf{f}_n)$, and $x = (0, \mathbf{f}_1)$, so that

$$\nu_m \geq \frac{\|\mathbf{1}_A + x\|}{\|\mathbf{1}_B + x\|} = \frac{m+1}{(m+1)^{\frac{1}{q}}} = (m+1)^{1/q'}.$$

From above, let $|A| \leq |B| = m$ and (x,y) have disjoint support with $A \cup B$. Then

$$\|(x,y) + \mathbf{1}_{\varepsilon A}\| \leq \|x\|_1 + \|y\|_q + |B|,$$

while if $k = |\operatorname{supp}(P_{\ell_1}(\mathbf{1}_B))|$, then

$$\|(x,y) + \mathbf{1}_{\eta B}\| = \|x\|_1 + k + (\|y\|_q^q + |B| - k)^{\frac{1}{q}} \geq \|x\|_1 + (\|y\|_q^q + |B|)^{\frac{1}{q}}.$$

So,

$$\frac{\|(x,y) + \mathbf{1}_{\varepsilon A}\|}{\|(x,y) + \mathbf{1}_{\eta B}\|} \leq \frac{\|x\|_1 + \|y\|_q + |B|}{\|x\|_1 + (\|y\|_q^q + |B|)^{\frac{1}{q}}} \leq \frac{\|y\|_q + |B|}{(\|y\|_q^q + |B|)^{\frac{1}{q}}},$$

and the latter is easily seen to be maximized at $\|y\|_q = 1$ and $|B| = m$. So $\nu_m \leq (1+m)^{\frac{1}{q'}}$ as asserted.

Remark 4.1.13. With similar (but slightly more tedious) computations one can show that, for $\ell_p \oplus c_0$, $1 < p < \infty$, one has

$$\nu_m = \widetilde{\mathbf{L}}_m = \mathbf{L}_m = 1 + m^{\frac{1}{p}},$$

while $\tilde{\mu}_m = \mu_m = 1 + (m-1)^{\frac{1}{p}}$, so again equality fails in (4.25).

Example 4.1.14 (A superdemocratic and not quasi-greedy basis)**.** Theorem 4.1.5 becomes asymptotically optimal when $\tilde{\mu}_m \approx 1$, as in this case $\mathbf{L}_m \approx k_m$ and $\widetilde{\mathbf{L}}_m \approx g_m$. We give a non-trivial example of this situation, which is a small variation of [31, Example 4.8]. This example has the additional interesting property of being unconditional with constant coefficients but not quasi-greedy. More precisely, for every $1 \leq q \leq \infty$, there exists $(\mathbb{X}, \mathscr{B})$ such that

a) $\nu_m \approx \tilde{\mu}_m \approx 1$,

b) $g_m \approx \tilde{g}_m \approx k_m \approx (\log m)^{1/q'}$,

c) $\mathbf{L}_m \approx \widetilde{\mathbf{L}}_m \approx (\log m)^{1/q'}$.

So, in this case Theorems 4.1.3 and 4.1.5 are asymptotically optimal.

Let $\mathscr{D}_k$ denote the set of all dyadic intervals $I \subset [0,1]$ with length $|I| = 2^{-k}$, and $\mathscr{D} = \cup_{k \geq 0} \mathscr{D}_k$. Consider the space $\mathfrak{f}_1^q$ of all (real) sequences $\mathbf{a} = (a_I)_{I \in \mathscr{D}}$ such that

$$\|\mathbf{a}\|_{\mathfrak{f}_1^q} = \Big\| \big[\sum_I |a_I \chi_I^{(1)}|^q \big]^{\frac{1}{q}} \Big\|_{L_1} < \infty,$$

where $\chi_I^{(1)} = |I|^{-1}\chi_I$. It is well known that $\{\mathbf{e}_I\}_{I\in\mathscr{D}}$, the canonical basis, is unconditional and democratic in $\mathfrak{f}_1^q$ (see [43, 52]). Moreover, for some $c_q \geq 1$ we have

$$\tfrac{1}{c_q}|A| \leq \|\mathbf{1}_{\varepsilon A}\|_{\mathfrak{f}_1^q} \leq |A|, \quad \forall A \subset \mathscr{D}, \ \varepsilon \in \Psi_A.$$

From the definition we also have

$$\Big\| \sum_k b_k 2^{-k} \mathbf{1}_{\mathscr{D}_k} \Big\|_{\mathfrak{f}_1^q} = \big(\sum_k |b_k|^q \big)^{\frac{1}{q}},$$

since $2^{-k}\sum_{\mathscr{D}_k} \chi_I^{(1)} = \chi_{[0,1]}$. For every $m \geq 1$ we shall pick a subset $\{k_1, \ldots, k_m\} \subset \mathbb{N}_0$, and look at the finite dimensional space F_m consisting of sequences supported in $\cup_{j=1}^m \mathscr{D}_{k_j}$. We order the canonical basis by $\cup_{j=1}^m \{\mathbf{e}_I\}_{I\in\mathscr{D}_{k_j}}$, so we may as well write their elements as $\mathbf{a} = (a_j)_{j=1}^{\dim F_m}$. We also consider in F_m the James norm

$$\|(a_j)\|_{J_q} = \sup_{m_0=0<m_1<\ldots} \Big[\sum_{k\geq 0} \big| \sum_{m_k<j\leq m_{k+1}} a_j \big|^q \Big]^{\frac{1}{q}}.$$

Note that $\|\mathbf{a}\|_{J_q} \leq \|\mathbf{a}\|_{\ell_1}$, with equality iff all the a_j's have the same sign[3]. In particular,

$$\|\mathbf{1}_A\|_{J_q} = |A|.$$

Now set in F_m a new norm

$$|||\mathbf{a}||| = \max\Big\{ \|\mathbf{a}\|_{\mathfrak{f}_1^q}, \ \|\mathbf{a}\|_{J_q} \Big\},$$

and observe that $(1/c_q)|A| \leq |||\mathbf{1}_{\varepsilon A}||| \leq |A|$, with c_q independent of m and k_j. Also, the vector $x = \sum_{j=1}^m (-1)^{j+1} 2^{-k_j} \mathbf{1}_{\mathscr{D}_{k_j}}$ has

$$\|x\|_{\mathfrak{f}_1^q} = \|x\|_{J_q} = |||x||| = m^{\frac{1}{q}}.$$

At this point we write $m = 2n$ and choose our k_j's as

$$k_{2j+1} = j \quad \text{and} \quad k_{2j+2} = n + j, \quad j = 0, \ldots, n-1.$$

Then if $P = \sum_{j \text{ odd}} 2^{k_j} = 2^n - 1$ we have $G_P(x) = \sum_{j \text{ odd}} 2^{-k_j} \mathbf{1}_{\mathscr{D}_{k_j}}$, which implies

$$\|G_P(x)\|_{\mathfrak{f}_1^q} = n^{\frac{1}{q}}, \quad \|G_P(x)\|_{J_q} = n, \quad \text{and} \quad |||G_P(x)||| = n.$$

Therefore

$$g_{2^n} \geq |||G_P(x)||| / |||x||| \geq n^{1-\frac{1}{q}}.$$

We turn to estimate the unconditionality constant k_m of the space F_m. Given $|A| = m$, we first claim that

$$\|P_A(x)\|_{\ell_1} \leq c'_q (\log|A|)^{1/q'} \|x\|_{\mathfrak{f}_1^q}. \tag{4.37}$$

[3]Note that $|a-b| < (a^q + b^q)^{\frac{1}{q}}$ if $a, b > 0$, so therefore, to maximize the norm, consecutive elements with different signs should be in different blocks of the James norm.

This is clear when $q=1$ (since $\mathfrak{f}_1^1=\ell_1$). When $q=\infty$, it is a consequence e.g. of [43, Remark 5.6] (since $\mathfrak{f}_1^\infty$ is a 1-space, in the terminology of [43, (2.8)]). Thus one derives (4.37) by complex interpolation. From here

$$\|P_A(x)\|_{J_q}\le \|P_A(x)\|_{\ell_1}\le c_q'\,(\log|A|)^{1/q'}\,\|\|x\|\|.$$

Also, $\|P_A(x)\|_{\mathfrak{f}_1^q}\le \|x\|_{\mathfrak{f}_1^q}\le \|\|x\|\|$. Hence, $k_m\le c_q'(\log m)^{1/q'}$.

Finally, we consider the space $\mathbb{X}=\oplus_{\ell_1}F_m$ with $\mathscr{B}$ the consecutive union of the natural bases in F_m. Then, for a finite set A and $\varepsilon\in\Psi_A$, if we write $A_m=A\cap F_m$,

$$\tfrac{1}{c_q}|A|\le \|\|\mathbf{1}_{\varepsilon A}\|\|=\sum_N \|\|\mathbf{1}_{\varepsilon A_m}\|\|\le |A|,$$

so $\mathscr{B}$ is super-democratic. We claim further that $\nu_m=O(1)$. Let $|A|\le|B|\le m$, $\varepsilon\in\Psi_A$, $\eta\in\Psi_B$ and $x\in\mathbb{X}$ have disjoint support with $A\cup B$. Assuming first that $\|\|x\|\|\ge 2|B|$, we have

$$\frac{\|\|\mathbf{1}_{\varepsilon A}+x\|\|}{\|\|\mathbf{1}_{\eta B}+x\|\|}\le\frac{\|\|\mathbf{1}_{\varepsilon A}\|\|+\|\|x\|\|}{\|\|x\|\|-\|\|\mathbf{1}_{\eta B}\|\|}\le\frac{3/2\|\|x\|\|}{1/2\|\|x\|\|}=3,$$

since $\|\|\mathbf{1}_{\varepsilon A}\|\|,\|\|\mathbf{1}_{\eta B}\|\|\le|B|\le\|\|x\|\|/2$. Otherwise we have $\|\|x\|\|\le 2|B|$, which implies

$$\frac{\|\|\mathbf{1}_{\varepsilon A}+x\|\|}{\|\|\mathbf{1}_{\eta B}+x\|\|}\le\frac{\|\|\mathbf{1}_{\varepsilon A}\|\|+\|\|x\|\|}{\sum_m\|\mathbf{1}_{\eta B_m}+x_m\|_{\mathfrak{f}_1^q}}\le\frac{3|B|}{\sum_m\|\mathbf{1}_{\eta B_m}\|_{\mathfrak{f}_1^q}}\le 3c_q,$$

since $\sum_m\|\mathbf{1}_{\eta B_m}\|_{\mathfrak{f}_1^q}\ge(c_q)^{-1}\sum_m|B_m|=|B|$, $B_m=B\cap F_m$ and x_m is the projection of x in $\mathrm{supp}(x)\cap F_m$. Thus $\nu_m\lesssim 1$ as asserted. Finally, if we denote by $k_m^{\mathbb{X}}$ (or $g_m^{\mathbb{X}}$) the unconditionality (or the quasi-greedy) parameter for the space $\mathbb{X}$, observe that $k_m^{\mathbb{X}}\le\max_m k_m^{F_m}\le c_q'(\log m)^{1/q'}$, while if $m=2n$ we have

$$g_{2^n}^{\mathbb{X}}\ge g_{2^n}^{F_N}\ge n^{1/q'}.$$

Remark 4.1.15. This example is the first one in the literature such that a non quasi-greedy basis satisfies the inequality (2.12), that is, $\exists\, c_1,c_2>0$ such that

$$c_1\min_A|a_n|\,\|\|\mathbf{1}_A\|\|\;\le\;\|\|\sum_A a_n\mathbf{e}_n\|\|\;\le\;c_2\max_A|a_n|\;\|\|\mathbf{1}_A\|\|,\tag{4.38}$$

for all finite sets A and all scalars a_n. Indeed, the right hand side is a consequence of $\nu_m\approx 1$ (or that the basis is unconditional for constant coefficients) and Lemma 1.3.1. The left hand inequality is true for the norm $\|\cdot\|_{\mathfrak{f}_1^q}$, and since $\|\mathbf{1}_A\|_{\mathfrak{f}_1^q}\approx|A|\approx\|\|\mathbf{1}_A\|\|$, it will also hold for the norm $\|\|\cdot\|\|$.

4.2 Connection between embeddings and Lebesgue-type parameters

In the above section, we have studied some different upper bounds of $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$ using other quantities easier to compute (see for example Theorem 4.1.5). The problem with these estimates is that they have multiplicative bounds which in some cases may not be asymptotically optimal.

One example where we have this problem is the trigonometric system $(e^{inx})_{n\in\mathbb{Z}}$ in $L_p(\mathbb{T})$ with $1<p<\infty$, where V. N. Temlyakov proved in [76] (see Example 4.3.3) that the parameters g_m, k_m and $\mathbf{L}_m$ grow as $m^{h(p)}$, where $h(p)=|1/2-1/p|$. Also, in the Subsection 4.3.3 we will show that $\tilde{\mu}_m$ has the same order. Hence, using Theorems 4.1.3 or 4.1.5 we do not obtain the exact bound of $\mathbf{L}_m$ for the trigonometric system.

Trying to find one bound for $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$ that works for any type of basis, we will study from a new point of view the estimation of $\mathbf{L}_m$ and $\widetilde{\mathbf{L}}_m$. This new point of view is based on some embeddings between our Banach space $\mathbb{X}$ and some weighted discrete Lorentz spaces. Concretely, for a non-negative weight $w=\{w(j)\}_{j=1}^{\infty}$, the spaces that we consider are:

- The discrete weighted Lorentz space: of a non-negative weight w and $0<r\leq\infty$,

$$\ell_w^r=\left\{\mathbf{s}\in c_0 : \|\mathbf{s}\|_{\ell_w^r}:=\sum_{j=1}^{\infty}[s_j^* w(j)]^r\frac{1}{j}<\infty\right\}, \tag{4.39}$$

  where $(s_j^*)_j$ is the non-increasing rearrangement of the sequence $(s_j)_{j=1}^{\infty}$.

- The discrete weighted Marcinkiewicz space:

$$m(w)=\left\{\mathbf{s}\in c_0 : \|\mathbf{s}\|_{m(w)}:=\sup_{k\in\mathbb{N}}\tfrac{w(k)}{k}\textstyle\sum_{j=1}^{k}s_j^*<\infty\right\}. \tag{4.40}$$

The relation between democracy functions and embeddings goes back to the papers in the topic [45, 47, 84] and a detailed study for quasi-greedy bases was given in [3]. Our approach is closer to that in [35, Proposition 3.6 and Corollary 3.7], where bounds for $\mathbf{L}_m$ are obtained for general bases under assumptions of the form $\ell_{q,\infty}\hookrightarrow\mathbb{X}\hookrightarrow\ell_{p,1}$, where $\ell_{p,r}$ are the classical (unweighted) Lorentz spaces.

Definition 4.2.1. Let $\mathscr{B}=(\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space. We say that a sequence space $\mathbb{S}$ *embeds into* $\mathbb{X}$ *via* $\mathscr{B}$ (with norm c), denoted $\mathbb{S}\overset{\mathscr{B},c}{\hookrightarrow}\mathbb{X}$, if for every $\mathbf{s}=\{s_n\}_{n=1}^{\infty}\in\mathbb{S}$, there exists a unique $x\in\mathbb{X}$ such that $\mathbf{e}_n^*(x)=s_n$ and it holds:

$$\|x\|\leq c\,\|\mathbf{s}\|_{\mathbb{S}}=c\,\|\{\mathbf{e}_j^*(x)\}_{j=1}^{\infty}\|_{\mathbb{S}}. \tag{4.41}$$

Similarly, we say that $\mathbb{X}$ *embeds into* $\mathbb{S}$ *via* $\mathscr{B}$ (with norm c), denoted $\mathbb{X}\overset{\mathscr{B},c}{\hookrightarrow}\mathbb{S}$, if

$$\|\{\mathbf{e}_j^*(x)\}_{j=1}^{\infty}\|_{\mathbb{S}}\leq c\,\|x\|,\quad x\in\mathbb{X}. \tag{4.42}$$

4.2.1 Classes of weights

To work with the spaces ℓ_w^r and $m(w)$, we need to define exactly what is a weight and some classes of weights.

A *weight* is any sequence $w=\{w(j)\}_{j=1}^{\infty}$ of non-negative numbers with $w(1)>0$. We use the following notation:

- $w>0$ for a *positive* weight, that is, $w(j)>0$ for all $j=1,2,\dots$

- $\mathbb{W}$ for the set of positive non-decreasing weights, that is,

$$0 < w(1) \le w(2) \le \dots$$

- $\mathbb{W}_{\mathrm{d}}$ is the subset of *doubling* weights, that is, $w \in \mathbb{W}$ with $w(2j) \le cw(j)$, for some $c \ge 1$ and all $j = 1, 2, \dots$

- $\mathbb{W}_{\mathrm{qc}}$ is the subset of *quasi-concave* weights, that is, $w \in \mathbb{W}$ with

$$\frac{w(j+1)}{j+1} \le \frac{w(j)}{j}, \quad j = 1, 2, \dots$$

- $\mathbb{W}_{\mathrm{co}}$ is the subset of all *concave* weights, that is, $w \in \mathbb{W}$ with

$$\Delta^2 w(j) = \Delta w(j) - \Delta w(j-1) \le 0, \quad \text{for } j = 2, 3, ..., \tag{4.43}$$

$\Delta w(j) := w(j) - w(j-1)$, $j = 1, 2, \dots$, and by convention we always set $w(0) = 0$. It is easy to see from the above definitions that

$$\mathbb{W}_{\mathrm{co}} \subset \mathbb{W}_{\mathrm{qc}} \subset \mathbb{W}_{\mathrm{d}} \subset \mathbb{W}.$$

Also, every $w \in \mathbb{W}_{\mathrm{qc}}$ has a *smallest concave majorant* $w^\sharp \in \mathbb{W}_{\mathrm{co}}$ with $w \le w^\sharp \le 2w$ (see [61]). Finally, notice that $\varphi_u, \varphi_u^* \in \mathbb{W}_{\mathrm{qc}}$ (see Lemma 1.2.2 of Chapter 1).

Associated with a weight w we consider the following weights:

- *summing weight:* $\widetilde{w}(m) = \sum_{j=1}^m \frac{w(j)}{j}$.
- *difference weight:* $\widehat{w}(j) = j\Delta w(j)$ (if $w \in \mathbb{W}$).
- *dual weight:* $w'(j) = j/w(j)$ (if $w > 0$).

It is elementary to verify the following identities:

$$\widehat{\widetilde{w}} = w, \qquad \widetilde{\widehat{w}} = w, \qquad (w')' = w. \tag{4.44}$$

Moreover, for every $w \in \mathbb{W}$,

$$w \in \mathbb{W}_{\mathrm{qc}} \iff \widetilde{w} \in \mathbb{W}_{\mathrm{co}} \iff \widehat{w} \le w \iff w' \in \mathbb{W}_{\mathrm{qc}}. \tag{4.45}$$

Finally observe that, if $w \in \mathbb{W}$, then

$$\widetilde{w}(m) \le w(m) \sum_{j=1}^m \frac{1}{j} \le w(m)(1 + \ln(m)). \tag{4.46}$$

Example 4.2.2.

(i) If $w(j) = [\ln(j+c)]^\gamma$, $\gamma > 0$, then $w \in \mathbb{W}_{\mathrm{co}}$ (for sufficiently large c) and

$$\widetilde{w}(j) \approx [\ln(j+1)]^{\gamma+1}, \qquad \widehat{w}(j) \approx [\ln(j+1)]^{\gamma-1}.$$

(ii) If $w(j) = j^{\alpha}[\ln(j+c)]^{\gamma}$, with $\alpha \in (0,1)$ and $\gamma \in \mathbb{R}$ (or with $\alpha = 1$ and $\gamma \le 0$), then $w \in \mathbb{W}_{\rm co}$ (for sufficiently large c) and $\widetilde{w} \approx \widehat{w} \approx w$.

One nice property that we will sometimes be interested is the following:

$$c_1 w(m) \;\le\; \widetilde{w}(m) \;\le\; c_2\, w(m), \quad m = 1,2,\dots \tag{4.47}$$

for some $c_1, c_2 > 0$. We shall call these weights *regular*. We now give some conditions under which (4.47) holds. The lower estimate holds trivially with $c_1 = 1$ when $w \in \mathbb{W}_{\rm qc}$. More generally, one has the following

Proposition 4.2.3. Let $w \in \mathbb{W}_{\rm d}$ with doubling constant c. Then

$$w(m) \;\le\; \tfrac{c}{\ln 2}\, \widetilde{w}(m), \quad m = 1,2,\dots \tag{4.48}$$

Moreover, $\widetilde{w} \in \mathbb{W}_{\rm d}$ with doubling constant bounded by $3c/2$.

Proof. If $m = 2n+1$,

$$\widetilde{w}(m) \ge \sum_{j=n+1}^{2n+1} \frac{w(j)}{j} \ge w(n+1) \sum_{j=n+1}^{2n+1} \frac{1}{j} \ge w(2n+1)\frac{\ln 2}{c}.$$

Arguing similarly when $m = 2n$ shows (4.48). Finally, the last assertion follows from

$$\widetilde{w}(2m) = \sum_{j=1}^{m} \frac{w(2j)}{2j} + \sum_{j=1}^{m} \frac{w(2j-1)}{2j-1} \le \tfrac{c}{2} \sum_{j=1}^{m} \frac{w(j)}{j} + c \sum_{j=1}^{m} \frac{w(j)}{j} \;=\; \tfrac{3c}{2}\, \widetilde{w}(m).$$

□

The upper bound in (4.47) requires some power growth in w, as shown in Example 4.2.2. This growth is typically quantified with the notion of *dilation index* (see [61]). To each $w > 0$, we associate two *dilation sequences* given by

$$\phi_w(M) = \inf_{k \ge 1} \frac{w(Mk)}{w(k)} \quad \text{and} \quad \Phi_w(M) = \sup_{k \ge 1} \frac{w(Mk)}{w(k)}, \quad M = 1,2,3,\dots \tag{4.49}$$

The *lower and upper dilation indices* associated with w are defined, respectively, by

$$i_w = \sup_{M>1} \frac{\ln(\phi_w(M))}{\ln M} \qquad \text{and} \qquad I_w = \inf_{M>1} \frac{\ln(\Phi_w(M))}{\ln M}. \tag{4.50}$$

For instance, for the weights w in Example 4.2.2 we have $i_w = I_w = 0$ in case (i), and $i_w = I_w = \alpha$ in case (ii). Observe also that $\phi_{w'}(M) = M/\Phi_w(M)$, so we always have

$$i_{w'} = 1 - I_w. \tag{4.51}$$

Proposition 4.2.4. Let $w \in \mathbb{W}$. Then $\quad \sup_{m \ge 1} \frac{\widetilde{w}(m)}{w(m)} < \infty \quad \Longleftrightarrow \quad i_w > 0.$

Proof. Assume first that $i_w > 0$. Then, for some integer $s_0 > 1$ we have $\lambda := \phi_w(s_0) > 1$. Suppose first that $m = s_0^n$ for some $n = 1,2,3,\ldots$. Then,

$$\begin{aligned}\widetilde{w}(m) &= \widetilde{w}(s_0^n) = w(1) + \sum_{k=0}^{n-1} \sum_{j=s_0^k+1}^{s_0^{k+1}} \frac{w(j)}{j} \\ &\leq w(1) + \sum_{k=0}^{n-1} w(s_0^{k+1}) \sum_{j=s_0^k+1}^{s_0^{k+1}} \frac{1}{j} \leq (1+\ln s_0) \sum_{k=0}^{n} w(s_0^k)\,. \end{aligned} \tag{4.52}$$

Now, by definition, $\phi_w(s_0) \leq w(s_0^{k+1})/w(s_0^k)$, and therefore

$$w(s_0^k) \leq \lambda^{-1} w(s_0^{k+1}) \leq \ldots \leq \lambda^{-(n-k)} w(s_0^n), \quad k = 0,1,\ldots,n.$$

Inserting this expression into (4.52) we obtain

$$\widetilde{w}(m) \leq \frac{1+\ln(s_0)}{1-\lambda^{-1}}\, w(s_0^n) \,=\, c\, w(m)\,.$$

For arbitrary $m > 1$, choose $n \in \mathbb{N}$ such that $s_0^{n-1} < m \leq s_0^n$. Then,

$$\widetilde{w}(m) = \widetilde{w}(s_0^{n-1}) + \sum_{j=s_0^{n-1}+1}^{m} \frac{w(j)}{j} \leq c w(s_0^{n-1}) + w(m)\ln(s_0) \lesssim w(m).$$

Conversely, assume that $i_w = 0$. Then $\phi_w(M) = 1$ for all $M \geq 2$. In particular, for each $M \geq 2$ there exists $k_M \in \mathbb{N}$ with $\frac{w(Mk_M)}{w(k)} \leq 2$, $\forall k \geq k_M$. Therefore

$$\widetilde{w}(Mk_M) \geq \sum_{k=k_M}^{Mk_M} \frac{w(k)}{k} \geq \frac{1}{2} w(Mk_M)\ln M,$$

leading to $\sup_m \frac{\widetilde{w}(m)}{w(m)} = \infty$.

□

Corollary 4.2.5. Let $w \in \mathbb{W}_{\mathrm{qc}}$. Then w' is regular if and only if $I_w < 1$.

Proof. First, $w \in \mathbb{W}$ already implies $w'(m) = m/w(m) \leq \sum_{j=1}^m 1/w(j) = \widetilde{(w')}(m)$. Next $w \in \mathbb{W}_{\mathrm{qc}}$ implies $w' \in \mathbb{W}$, and by Proposition 4.2.4, the converse inequality $\widetilde{(w')} \lesssim w'$ is equivalent to $i_{w'} > 0$, and the result follows from the identity in (4.51). □

4.2.2 The discrete weighted Lorentz space ℓ_w^1

We remind the definition of the space ℓ_w^r: for a non-negative weight w and $0 < r \leq \infty$,

$$\ell_w^r = \Big\{ \mathbf{s} \in c_0 : \|\mathbf{s}\|_{\ell_w^r} := \sum_{j=1}^{\infty} [s_j^* w(j)]^r \frac{1}{j} < \infty \Big\}, \tag{4.53}$$

where $(s_j^*)_j$ is the non-increasing rearrangement of the sequence $(s_j)_{j=1}^\infty$. In the literature ℓ_w^r is sometimes denoted $d(r,\eta)$ with $\eta_j = \frac{w(j)^r}{j}$, and the weight η is required to decrease to 0 and $\sum_{j=1}^\infty \eta_j = \infty$ (see [63, p. 175] or references in [26, p. 28]). We will be dealing only with the case $r = 1$ but we shall consider more general weights, namely $\eta = \{w(j)/j\}$ and $\eta = \Delta w$, for $w \in \mathbb{W}$. Moreover, it is well-known that $d(1,\eta)$ are quasi-normed spaces if and only if $W(m) = \sum_{j=1}^m \eta_j$ satisfies a doubling condition (see [26, Theorem 2.2.16]).

Clearly if $w \in \mathbb{W}_{\mathrm{qc}}$ then $\widehat{w} \le w$ and therefore $\ell_w^1 \hookrightarrow \ell_{\widehat{w}}^1$. Below we show that this is the case also for $w \in \mathbb{W}_{\mathrm{d}}$. The following basic lemma will be used often.

Lemma 4.2.6. If ν, ξ are non-negative sequences, the following holds:

$$\tilde{\nu} \le \tilde{\xi} \quad \Longleftrightarrow \quad \sum_{j=1}^\infty a_j^* \frac{\nu(j)}{j} \le \sum_{j=1}^\infty a_j^* \frac{\xi(j)}{j}, \quad \forall \text{ non-increasing } a_j^*.$$

In particular, $\tilde{\nu} \le \tilde{\xi}$ if and only if $\ell_\xi^1 \hookrightarrow \ell_\nu^1$ with embedding of norm 1.

For the proof of this lemma, we use the following elementary identity known as Abel summation formula: for all finite sequences $\{x_n\}_{n=1}^N$ in $\mathbb{X}$ and $\{d_n\}_{n=1}^N$ in $\mathbb{K}$ it holds

$$x_1 d_1 + \sum_{n=2}^N d_n(x_n - x_{n-1}) = \sum_{n=1}^{N-1} (d_n - d_{n+1}) x_n + x_N d_N. \tag{4.54}$$

Proof. Suppose that $\tilde{\nu} \le \tilde{\xi}$. Then, using the Abel summation formula in (4.54),

$$\begin{aligned}
\sum_{j=1}^N a_j^* \frac{\nu(j)}{j} &= a_1^* \tilde{\nu}(1) + \sum_{j=2}^N a_j^* (\tilde{\nu}(j) - \tilde{\nu}(j-1)) = \sum_{j=1}^{N-1} (a_j^* - a_{j+1}^*) \tilde{\nu}(j) + a_N^* \tilde{\nu}(N) \\
&\le \sum_{j=1}^{N-1} (a_j^* - a_{j+1}^*) \tilde{\xi}(j) + a_N^* \tilde{\xi}(N) = \sum_{j=1}^N a_j^* \frac{\xi(j)}{j}.
\end{aligned}$$

Now, let $N \to \infty$ and we obtain the result. To show the other implication, we only have to take $a_j^* = 1$ if $1 \le j \le N$ and $a_j^* = 0$ if $j > N$. □

Corollary 4.2.7. (i) If $w \in \mathbb{W}_{\mathrm{d}}$, then $\ell_w^1 \hookrightarrow \ell_{\widehat{w}}^1$.

(ii) If $w \in \mathbb{W}$, then $\quad \ell_{\widehat{w}}^1 \hookrightarrow \ell_w^1 \quad \Longleftrightarrow \quad i_w > 0.$

(iii) If $w \in \mathbb{W}_{\mathrm{d}}$, then $\quad \ell_w^1 = \ell_{\widehat{w}}^1 \quad \Longleftrightarrow \quad i_w > 0.$

Proof. The proof follows combining Lemma 4.2.6 with (4.44) and Propositions 4.2.3 and 4.2.4. □

Corollary 4.2.8. $\ell_w^1 = c_0$ if and only if $\widetilde{w}$ is bounded. In particular, $\ell_{\widehat{w}}^1 = c_0$ if and only if $w \in \mathbb{W}$ is bounded.

Proof. The inclusion $\ell_w^1 \hookrightarrow c_0$ is always true. For the converse, write $c_0 = \ell_\xi^1$ with $\xi = \{1,0,0,\ldots\}$, and use Lemma 4.2.6. □

Theorem 4.2.9. Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be semi-normalized and biorthogonal in $\mathbb{X}$, that is, (ii) and (iv) of Definition 1.1.1. Let $w \in \mathbb{W}$. Then, the following are equivalent:

i) $\|\mathbf{1}_{\varepsilon A}\| \le w(|A|)$ for all finite $A \subset \mathbb{N}$ and all $\varepsilon \in \Psi_A$.

ii) $\|\sum a_n \mathbf{e}_n\| \le \|\mathbf{a}\|_{\ell^1_{\widehat{w}}}$, for all $\mathbf{a} = \{a_n\} \in c_{00}$.

Moreover, if $\mathscr{B}^*$ is total (condition (iii) of Definition 1.1.1) then each of the above is equivalent to

iii) $\ell^1_{\widehat{w}} \overset{\mathscr{B},1}{\hookrightarrow} \mathbb{X}$.

As noted above, $\ell^1_{\widehat{w}}$ is a linear space if and only if the sequence w is doubling.

Proof of Theorem 4.2.9. The implication ii) $\Rightarrow$ i) is clear since

$$\|\mathbf{1}_{\varepsilon A}\| \le \left\|\{\varepsilon_j\}_{j\in A}\right\|_{\ell^1_{\widehat{w}}} = \sum_{j=1}^{|A|} \Delta w(j) = w(|A|).$$

We now show that i) $\Rightarrow$ ii). Let $\mathbf{a} \in c_{00}$ and $N = |\operatorname{supp}(\mathbf{a})|$. Write $a_j^* = |a_{\pi(j)}|$, where $\pi : \{1,...,N\} \to \operatorname{supp}(\mathbf{a})$ is a greedy bijection, that is $|a_{\pi(j)}| \ge |a_{\pi(j+1)}|$, $j = 1,2,...$ Let also $\varepsilon_j = \operatorname{sign}(a_{\pi(j)})$. If we define

$$S_J = \sum_{j=1}^{J} \varepsilon_j \mathbf{e}_{\pi(j)} \tag{4.55}$$

(and $S_0 = 0$), then using Abel summation formula (4.54) we can write

$$\sum_{n\in\operatorname{supp}(\mathbf{a})} a_n \mathbf{e}_n = \sum_{j=1}^{N} a_j^* \varepsilon_j \mathbf{e}_{\pi(j)} = \sum_{j=1}^{N} a_j^*(S_j - S_{j-1}) = \sum_{j=1}^{N-1} (a_j^* - a_{j+1}^*) S_j + a_N^* S_N.$$

Then, by assumption i),

$$\begin{aligned}\Big\|\sum_{n\in\operatorname{supp}(\mathbf{a})} a_n \mathbf{e}_n\Big\| &\le \sum_{j=1}^{N-1} (a_j^* - a_{j+1}^*)\|S_j\| + a_N^*\|S_N\| \le \sum_{j=1}^{N-1} (a_j^* - a_{j+1}^*) w(j) + a_N^* w(N) \\ &= a_1^* w(1) + \sum_{j=2}^{N} a_j^*(w(j) - w(j-1)) = \|\mathbf{a}\|_{\ell^1_{\widehat{w}}}, \end{aligned} \tag{4.56}$$

which is the desired result.

The implication iii) $\Rightarrow$ ii) is immediate, so it remains to prove i) $\Rightarrow$ iii) under the assumption that the system is total. Let $\mathbf{a} \in \ell^1_{\widehat{w}}$, which we shall assume with infinite support (otherwise the result follows from ii)). As before, write $a_j^* = |a_{\pi(j)}|$ where $\pi : \mathbb{N} \to \operatorname{supp}(\mathbf{a})$ is a greedy bijection, and $\varepsilon_j = \operatorname{sign}(a_{\pi(j)})$. Letting S_J be as in (4.55), we have

$$\begin{aligned}
\sum_{j=1}^{J} (a_j^* - a_{j+1}^*)\|S_j\| &\leq \sum_{j=1}^{J} (a_j^* - a_{j+1}^*) w(j) \\
\text{(by (4.54))} \quad &= a_1^* w(1) + \sum_{j=2}^{J+1} a_j^* [w(j) - w(j-1)] - a_{J+1}^* w(J+1) \\
&\leq \sum_{j=1}^{\infty} a_j^* \Delta w(j) = \|\mathbf{a}\|_{\ell^1_{\widehat{w}}} < \infty.
\end{aligned}$$

Therefore, the series $\sum_{j=1}^{\infty} (a_j^* - a_{j+1}^*) S_j$ converges to some $x \in \mathbb{X}$ and $\|x\| \leq \|\mathbf{a}\|_{\ell^1_{\widehat{w}}}$. It only remains to show that

$$\mathbf{e}_n^*(x) = a_n, \quad \forall\, n \in \mathbb{N}. \tag{4.57}$$

If $n \notin \operatorname{supp}(\mathbf{a})$ then $\mathbf{e}_n^*(S_j) = 0$ for all j, and thus $\mathbf{e}_n^*(x) = 0$. Let then $n \in \operatorname{supp}(\mathbf{a})$, and write $n = \pi(j_n)$, so that

$$\begin{aligned}
\mathbf{e}_n^*(x) &= \lim_{J \to \infty} \mathbf{e}_n^* \left(\sum_{j=1}^{J} (a_j^* - a_{j+1}^*) S_j \right) = \lim_{J \to \infty} \sum_{j=j_n}^{J} (a_j^* - a_{j+1}^*) \varepsilon_{j_n} \\
&= \lim_{J \to \infty} (a_{j_n}^* - a_{J+1}^*) \varepsilon_{j_n} = a_{j_n}^* \varepsilon_{j_n} = a_{\pi(j_n)} = a_n,
\end{aligned}$$

where we have used that $\mathbf{a} \in c_0$. Finally, there is a unique element x with the property (4.57) by the totality of the system $\mathscr{B}^*$. This shows that $\ell^1_{\widehat{w}} \overset{\mathscr{B},1}{\hookrightarrow} \mathbb{X}$, and completes the proof of the theorem. □

Remark 4.2.10. The statement of Theorem 4.2.9 resembles a well known property of the classical Lorentz spaces $L_{p,1}$. Namely, if $\|\cdot\|$ is an order preserving norm defined on the set $\mathscr{S}$ of all simple functions of a measure space $(\Omega, \Sigma, d\mu)$, then the inequality $\|\chi_E\| \leq \mu(E)^{1/p}$ for all $E \in \Sigma$, implies that $\|f\| \leq \|f\|_{L_{p,1}(\mu)}$ for all $f \in \mathscr{S}$ (see [75, Thm V.3.11]).

Remark 4.2.11. In the special setting of quasi-greedy bases, a result similar to Theorem 4.2.9 was proved earlier in [49, Lemma 2.1]. More precisely, if $\mathscr{B}$ is quasi-greedy in $\mathbb{X}$ and $w \in \mathbb{W}_{\mathrm{d}}$ is such that $\|\mathbf{1}_A\| \leq w(|A|)$, then $\ell^1_w \hookrightarrow \mathbb{X}$ via $\mathscr{B}$. Theorem 4.2.9 actually shows that one can choose a better space, since $\ell^1_w \subset \ell^1_{\widehat{w}}$. See also [3, Theorem 3.1].

4.2.3 The discrete weighted Marcinkiewicz space $m(w)$

We now turn to the *discrete weighted Marcinkiewicz space* defined in (4.40), which we compare with the Lorentz space ℓ^∞_w in (4.39), where

$$\|\mathbf{s}\|_{\ell^\infty_w} = \sup_j w(j) s_j^*.$$

First, we recall the definition of $m(w)$: for a non-negative weight w,

$$m(w) = \left\{ \mathbf{s} \in c_0 : \|\mathbf{s}\|_{m(w)} := \sup_{k \in \mathbb{N}} \tfrac{w(k)}{k} \textstyle\sum_{j=1}^{k} s_j^* < \infty \right\}. \tag{4.58}$$

Observe that, for $\mathbf{s} = \{s_n\} \in c_0$,

$$\|\mathbf{s}\|_{m(w)} = \sup_{\substack{|A|=m \\ m\in\mathbb{N}}} \frac{w(m)}{m} \sum_{n\in A} |s_n|. \tag{4.59}$$

So $m(w)$ is a normed space for any non-negative weight w. It is also easy to see that $m(w) = c_0$ if and only if w is bounded.

Lemma 4.2.12. (i) $m(w) \hookrightarrow \ell_w^\infty$, with embedding norm 1.

(ii) If $w \in \mathbb{W}$, then $\ell_w^\infty \overset{c}{\hookrightarrow} m(w)$ if and only if $\widetilde{(w')} \leq cw'$, that is

$$\sum_{j=1}^{m} \frac{1}{w(j)} \leq \frac{c\,m}{w(m)}, \quad m = 1, 2, \dots \tag{4.60}$$

(iii) If $w \in \mathbb{W}_{\mathrm{qc}}$, then $\quad m(w) = \ell_w^\infty \quad \Longleftrightarrow \quad I_w < 1.$

Proof. (i) This follows easily from $s_m^* \leq \frac{1}{m}\sum_{j=1}^m s_j^*$.

(ii) Assume that $\ell_w^\infty \overset{c}{\hookrightarrow} m(w)$. Then picking $\mathbf{s} = \{1/w(j)\} \in \ell_w^\infty$ we obtain

$$\|\mathbf{s}\|_{m(w)} = \sup_m \frac{w(m)}{m} \sum_{j=1}^{m} \frac{1}{w(j)} \leq c\|\mathbf{s}\|_{\ell_w^\infty} = c,$$

obtaining the condition (4.60). Conversely, (4.60) and the inequality $s_j^* \leq \frac{\|\mathbf{s}\|_{\ell_w^\infty}}{w(j)}$ easily lead to $\|\mathbf{s}\|_{m(w)} \leq c\|\mathbf{s}\|_{\ell_w^\infty}$.

(iii) This follows from (i), (ii) and Corollary 4.2.5. □

Now, we show a result studying the relation between ℓ_w^1 and $m(w)$. This is a duality result which is known in the literature (see [26, §2.4]) but we present an elementary proof.

Theorem 4.2.13. If $w \in \mathbb{W}_{\mathrm{d}}$ and $\inf\limits_{m\in\mathbb{N}} \frac{w(m)}{m} = 0$, then $(\ell_{\widehat{w}}^1)^* = m(w')$ isometrically.

Proof. Let $\mathbf{a} \in \ell_{\widehat{w}}^1$ and $\mathbf{b} \in m(w')$. We may apply Lemma 4.2.6 with $\nu(j) = jb_j^*$ and $\xi(j) = \|\mathbf{b}\|_{m(w')}\, \widehat{w}(j)$, since $\widetilde{\nu}(n) \leq \widetilde{\xi}(n)$, and conclude that

$$\sum_{j=1}^{\infty} |a_j b_j| \;\leq\; \sum_{j=1}^{\infty} a_j^* b_j^* \leq \|\mathbf{b}\|_{m(w')} \sum_{j=1}^{\infty} a_j^* \Delta w(j) \;=\; \|\mathbf{b}\|_{m(w')} \|\mathbf{a}\|_{\ell_{\widehat{w}}^1}. \tag{4.61}$$

This shows that $m(w') \subseteq (\ell_{\widehat{w}}^1)^*$. Conversely let $h \in (\ell_{\widehat{w}}^1)^*$ and denote $b_n = h(\mathbf{e}_n)$ with $(\mathbf{e}_n)_n$ the standard basis in c_{00}. If $\varepsilon_n = \operatorname{sign}(b_n)$ and $\bar{\varepsilon}_n$ is the conjugate sign, then for each $|A| = m$ we have

$$\sum_{n\in A} |b_n| = |\sum_{n\in A} \bar{\varepsilon}_n b_n| \leq \|h\| \|\mathbf{1}_{\bar{\varepsilon}A}\|_{\ell_{\widehat{w}}^1} = \|h\|\, w(m). \tag{4.62}$$

We claim that $\mathbf{b} = \{b_n\}_{n=1}^\infty \in c_0$. Indeed, if not, there would be a $\delta > 0$ and a subsequence $|b_{n_j}| \geq \delta$, and (4.62) gives $\delta m \leq \sum_{j=1}^m |b_{n_j}| \leq \|h\| w(m)$, which contradicts that $\inf_m \frac{w(m)}{m} = 0$.

Finally, (4.62) implies that $\sum_{j=1}^m b_j^* \le \|h\| w(m)$, and therefore $\|\mathbf{b}\|_{m(w')} \le \|h\|$. This completes the proof. $\square$

Theorem 4.2.14. Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be semi-normalized biorthogonal and complete system in $\mathbb{X}$, that is, (i), (ii) and (iv) of Definition 1.1.1, and w a positive sequence. Then, the following are equivalent:

(i) $\|\mathbf{1}_{\varepsilon A}^*\|_* \le w(|A|)$ for all finite $A \subset \mathbb{N}$ and all $\varepsilon \in \Psi_A$.

(ii) $\mathbb{X} \overset{\mathscr{B},1}{\hookrightarrow} m(w')$, with $w' = \{j/w(j)\}_{j=1}^\infty$.

Proof of Theorem 4.2.14. First, we show (i) $\Rightarrow$ (ii). For $x \in \mathbb{X}$, write $a_j^*(x) = |\mathbf{e}_{\pi(j)}^*(x)|$, where π is a greedy permutation onto $\operatorname{supp}(x)$, that is, $|\mathbf{e}_{\pi(j)}^*(x)| \ge |\mathbf{e}_{\pi(j+1)}^*(x)|$, $j = 1, 2, \ldots$. We also let $\varepsilon_j = \operatorname{sign}(\mathbf{e}_{\pi(j)}^*(x))$, $j = 1, 2\ldots$ Then

$$\begin{aligned}
\frac{w'(m)}{m} \sum_{j=1}^m a_j^*(x) &= \frac{1}{w(m)} \sum_{j=1}^m a_j^*(x) = \frac{1}{w(m)} \Big(\sum_{j=1}^N \bar{\varepsilon}_j \mathbf{e}_{\pi(j)}^* \Big)(x) \\
&\le \frac{1}{w(m)} \left\| \sum_{j=1}^m \bar{\varepsilon}_j \mathbf{e}_{\pi(j)}^* \right\|_* \|x\| \le \|x\|.
\end{aligned}$$

Now, we prove (ii) $\Rightarrow$ (i). Let $A \subset \mathbb{N}$ be a finite set and $\varepsilon \in \Psi_A$. Then,

$$\|\mathbf{1}_{\varepsilon A}^*\|_* = \sup_{\|x\|=1} |\mathbf{1}_{\varepsilon A}^*(x)| = \sup_{\|x\|=1} \Big| \sum_{j \in A} \varepsilon_j \mathbf{e}_j^*(x) \Big|. \tag{4.63}$$

Given $x \in \mathbb{X}$, if we take $\{a_j^*(x)\}_j$ as in the proof of the previous implication, we have

$$\left| \sum_{j \in A} \varepsilon_j \mathbf{e}_j^*(x) \right| \le \sum_{j \in A} |\mathbf{e}_j^*(x)| \le \sum_{j=1}^{|A|} a_j^*(x) \le w(|A|) \|x\|,$$

with the last inequality due to the assumption (ii). Inserting this estimate into (4.63) gives the desired expression (i). $\square$

Remark 4.2.15. In the setting of quasi-greedy bases, a different version of Theorem 4.2.14 involving the unsigned lower democracy function h_l was proved in [49, Lemma 2.2]. Namely, $\mathbb{X} \hookrightarrow \ell_{h_l}^\infty$ (see also [3, Theorem 3.1]). Such embedding, however, cannot hold for general bases. For instance, consider the space $\mathbb{X}$ of all sequences $\mathbf{a} = \{a_n\}_{n=1}^\infty \in c_0$ with

$$\|\mathbf{a}\| := \sup_{M \ge 1} \left| \sum_{n=1}^M a_n \right| < \infty,$$

with the standard canonical basis $(\mathbf{e}_n)_n$. Then, $h_l(m) = m$. However, the embedding $\mathbb{X} \hookrightarrow \ell_{h_l}^\infty$ cannot hold since $\mathbf{a} = \{(-1)^n\}_{n=1}^m$ belongs to $\mathbb{X}$ with $\|\mathbf{a}\| = 1$, but $\sup_n n a_n^* \to \infty$.

4.2.4 Estimate of the Lebesgue-type parameters for the TGA

In this section, our purpose is to prove a result that connects the parameters $\widetilde{\mathbf{L}}_m$ and $\mathbf{L}_m$ and the corresponding dual parameters with the fundamental functions φ_u and φ_u^*. In general, for a quantity $Q_m = Q_m[\mathscr{B},\mathbb{X}]$, $Q_m^* = Q_m[\mathscr{B}^*,\mathbb{Y}]$ with $\mathbb{Y} = [\mathscr{B}^*]$. We need the following notation: for two general weights w_1, w_2, we define

$$T_m(w_1,w_2) \quad := \quad \sum_{j=1}^{m} \frac{w_1(j)}{j} \Delta w_2(j), \tag{4.64}$$

$$\overline{T}_m(w_1,w_2) \quad := \quad \min\big\{T_m(w_1,w_2), T_m(w_2,w_1)\big\}, \tag{4.65}$$

and

$$S_m(w_1,w_2) \quad := \quad \sum_{j=1}^{m} \Delta w_1(j) \Delta w_2(j). \tag{4.66}$$

Theorem 4.2.16. Let $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^{\infty}$ be semi-normalized, complete and biorthogonal system in $\mathbb{X}$. Let $\overline{T}_m = \overline{T}_m(\varphi_u,\varphi_u^*)$. Then the following hold:

$$k_m \le \overline{T}_m, \qquad \mathbf{L}_m, \mathbf{L}_m^* \le 1 + 3\overline{T}_m, \quad \text{and} \quad \widetilde{\mathbf{L}}_m, \widetilde{\mathbf{L}}_m^* \le 1 + 2\overline{T}_m. \tag{4.67}$$

If, additionally, φ_u (resp. φ_u^*) is concave, then $S_m = S_m(\varphi_u,\varphi_u^*) \le \overline{T}_m$ and

$$k_m \le S_m, \quad \mathbf{L}_m \le 1 + 3S_m, \quad \widetilde{\mathbf{L}}_m \le 1 + 2S_m, \tag{4.68}$$

(respectively, for $k_m^*, \mathbf{L}_m^*, \widetilde{\mathbf{L}}_m^*$).

Finally, these estimates are best possible, in the sense that there exist $\mathbb{X}$ and $\{\mathbf{e}_n,\mathbf{e}_n^*\}$ for which all the equalities hold.

Throughout this section, the system $\{\mathbf{e}_n,\mathbf{e}_n^*\}_{n=1}^{\infty}$ is an in the statement of Theorem 4.2.16, and the sequences $w_1, w_2 \in \mathbb{W}$ are such that for any finite set $|A| \le m$ and $\varepsilon \in \Psi_A$,

$$(1) \quad \|\mathbf{1}_{\varepsilon A}\| \le w_1(m) \qquad \text{and} \qquad (2) \quad \|\mathbf{1}_{\varepsilon A}^*\|_* \le w_2(m). \tag{4.69}$$

Clearly, these inequalities are satisfied for $w_1 = \varphi_u$ and $w_2 = \varphi_u^*$.

<u>**Estimates for k_m:**</u>

Lemma 4.2.17. Suppose the sequences $w_1, w_2 \in \mathbb{W}$ satisfy (4.69). Then:

(i) If $w_1 \in \mathbb{W}_{\mathrm{qc}}$, then $k_m \le \overline{T}_m(w_1,w_2)$.

(ii) If $w_1 \in \mathbb{W}_{\mathrm{co}}$, then $k_m \le S_m(w_1,w_2)$.

Proof. Given any $x \in \mathbb{X}$, we denote by $\{a_j^*(x)\}$ the non-increasing rearrangement of $\big(\mathbf{e}_j^*(x)\big)_j$, that is, $a_j^*(x) = |\mathbf{e}_{\pi(j)}^*(x)|$, where π is a greedy bijection onto $\mathrm{supp}(x)$ (see the proof of Theorem 4.2.14). If $|A| \le m$ and $\varepsilon \in \Psi_A$, then part (1) of (4.69) and the implication i) $\Rightarrow$ ii) of Theorem

4.2.9 imply

$$\|P_A(x)\| \leq \sum_{j=1}^{|A|} a_j^*(P_A(x))\Delta w_1(j) \leq \sum_{j=1}^{m} a_j^*(x)\Delta w_1(j) =: A_m(x). \tag{4.70}$$

We start by proving (ii). Denoting $S_J(x) = \sum_{j=1}^{J} a_j^*(x)$, and using the Abel summation formula (4.54)

$$\begin{aligned} A_m(x) &= S_1(x)\, w_1(1) + \sum_{j=2}^{m} [S_j(x) - S_{j-1}(x)]\Delta w_1(j) \\ &= \sum_{j=1}^{m-1} \Big[\Delta w_1(j) - \Delta w_1(j+1)\Big] S_j(x) + \Delta w_1(m) S_m(x). \end{aligned} \tag{4.71}$$

Now, the inequality (2) in (4.69) and i) $\Rightarrow$ ii) of Theorem 4.2.14 imply that

$$\frac{1}{w_2(j)} S_j(x) = \frac{w_2'(j)}{j} \sum_{k=1}^{j} a_k^*(x) \leq \|x\|, \quad j = 1, 2, \ldots \tag{4.72}$$

Since $w_1 \in \mathbb{W}_{\rm co}$, we may insert in (4.71) the inequalities for $S_j(x)$ in (4.72), and then another use of the Abel summation formula (4.54) gives,

$$\begin{aligned} A_m(x) &\leq \Big[\sum_{j=1}^{m-1} [\Delta w_1(j) - \Delta w_1(j+1)] w_2(j) + \Delta w_1(m) w_2(m)\Big] \|x\| \\ &= \sum_{j=1}^{m} \Delta w_1(j)\Delta w_2(j)\ \|x\| = S_m(w_1, w_2)\ \|x\|. \end{aligned} \tag{4.73}$$

Plug this into (4.70) to obtain the desired estimate for k_m.

To prove (i) assume $w_1 \in \mathbb{W}_{\rm qc}$. Then $w_1 \leq \widetilde{w_1}$, so that (4.69) holds with w_1 replaced by $\widetilde{w_1}$. Since $\widetilde{w_1} \in \mathbb{W}_{\rm co}$ (see (4.45)), by part (ii) of this Lemma (just proved)

$$k_m \leq S_m(\widetilde{w_1}, w_2) = \sum_{j=1}^{m} \frac{w_1(j)}{j} \Delta w_2(j) = T_m(w_1, w_2).$$

On the other hand, observe that in (4.70) we could also argue as follows:

$$\begin{aligned} A_m(x) &= \sum_{j=1}^{m} a_j^*(x)\Delta w_1(j) \leq \sup_{k\in\mathbb{N}} \big[\tfrac{k}{w_2(k)} a_k^*(x)\big] \sum_{j=1}^{m} \frac{w_2(j)}{j} \Delta w_1(j) \\ &\leq \|\{a_j^*(x)\}\|_{m(w_2')}\, T_m(w_2, w_1) \ \leq\ \|x\|\, T_m(w_2, w_1), \end{aligned}$$

the last inequality due to (2) in (4.69) and (i)$\Rightarrow$(ii) in Theorem 4.2.14. Thus, we have shown that (4.69) implies

$$k_m \leq \min\big\{T_m(w_1, w_2), T_m(w_2, w_1)\big\} =: \overline{T}_m(w_1, w_2).$$

$\square$

Estimates for $\mathbf{L}_m$:

Lemma 4.2.18. Suppose the sequences $w_1, w_2 \in \mathbb{W}$ satisfy (4.69). Then:

(i) If $w_1 \in \mathbb{W}_{\rm qc}$, then $\mathbf{L}_m \le 1 + 3\overline{T}_m(w_1, w_2)$.

(ii) If $w_1 \in \mathbb{W}_{\rm co}$, then $\mathbf{L}_m \le 1 + 3S_m(w_1, w_2)$.

Proof. We follow the standard approach in [60]. Let $x \in \mathbb{X}$ and write $G_m(x) = P_\Gamma(x)$ for some $\Gamma \in \mathscr{G}(x,m)$. Take any $z = \sum_{n\in B} c_n \mathbf{e}_n$ with $|B| \le m$. Then,

$$\begin{aligned} \|x - G_m(x)\| &= \|x - P_{B\cup\Gamma}(x) + P_{B\setminus\Gamma}(x)\| \\ &\le \|P_{(B\cup\Gamma)^c}(x)\| + \|P_{B\setminus\Gamma}(x)\| =: I + II. \end{aligned} \tag{4.74}$$

For the first term we use that $P_{(B\cup\Gamma)^c}(x) = P_{(B\cup\Gamma)^c}(x-z)$, and therefore

$$\begin{aligned} I = \|(\mathrm{Id}_{\mathbb{X}} - P_{B\cup\Gamma})(x-z)\| &\le \|x-z\| + \|P_B(x-z)\| + \|P_{\Gamma\setminus B}(x-z)\| \\ &\le (1+2k_m)\|x-z\|. \end{aligned} \tag{4.75}$$

To estimate II we proceed as follows. First, using (4.69) and i) $\Rightarrow$ ii) in Theorem 4.2.9,

$$II = \|P_{B\setminus\Gamma}(x)\| \le \sum_{j=1}^{|B\setminus\Gamma|} a_j^*(P_{B\setminus\Gamma}(x))\Delta w_1(j) \le \sum_{j=1}^{|\Gamma\setminus B|} a_j^*(P_{\Gamma\setminus B}(x))\Delta w_1(j),$$

where in the last step we have used that Γ is a greedy set for x and $|B\setminus\Gamma| \le |\Gamma\setminus B|$. Now, $P_{\Gamma\setminus B}(x) = P_{\Gamma\setminus B}(x-z)$, and we may use that $a_j^*(P_{\Gamma\setminus B}(x-z)) \le a_j^*(x-z)$ to conclude

$$II = \|P_{B\setminus\Gamma}(x)\| \le \sum_{j=1}^{|\Gamma\setminus B|} a_j^*(x-z)\Delta w_1(j).$$

The right hand side resembles that of (4.70), with $A_m(x)$ replaced by $A_{|\Gamma\setminus B|}(x-z)$. We estimate $A_{|\Gamma\setminus B|}(x-z)$ as in Lemma 4.2.17. For $w_1 \in \mathbb{W}_{\rm co}$ (case (ii)), we obtain

$$II = \|P_{B\setminus\Gamma}(x)\| \le S_m(w_1,w_2)\,\|x-z\|. \tag{4.76}$$

Thus, combining (4.74), (4.75), and (4.76), together with (ii) of Lemma 4.2.17, we are led to

$$\begin{aligned} \|x - G_m(x)\| &\le (1 + 2k_m + S_m(w_1,w_2))\,\|x-z\| \\ &\le (1 + 3\,S_m(w_1,w_2))\|x-z\|. \end{aligned}$$

Taking the infimum over all such z we finally obtain $\mathbf{L}_m \le 1 + 3\,S_m(w_1,w_2)$.

For $w_1 \in \mathbb{W}_{\rm qc}$ (case (i)), we modify the preceding argument (as we did in the proof of Lemma 4.2.17 (i)) to obtain $\mathbf{L}_m \le 1 + 3\,\overline{T}_m(w_1,w_2)$. □

Estimates for $\widetilde{\mathbf{L}}_m$:

Lemma 4.2.19. Suppose the sequences $w_1, w_2 \in \mathbb{W}$ satisfy (4.69). Then:

(i) If $w_1 \in \mathbb{W}_{\rm qc}$, then $\widetilde{\mathbf{L}}_m \le 1 + 2\overline{T}_m(w_1,w_2)$.

(ii) If $w_1 \in \mathbb{W}_{\rm co}$, then $\widetilde{\mathbf{L}}_m \le 1 + 2S_m(w_1, w_2)$.

Proof. Repeat the argument from the preceding lemma with $z = P_B(x)$. The term II is estimated exactly as above, while for the term I we proceed as follows

$$\begin{aligned} I = \|(\mathrm{Id}_{\mathbb{X}} - P_{B\cup\Gamma})(x)\| &= \|x - P_B(x) - P_{\Gamma\setminus B}(x - P_B(x))\| \\ &\le \|x - P_B(x)\| + \|P_{\Gamma\setminus B}(x - P_B(x))\| \\ &\le (1 + k_m)\|x - P_B(x)\|. \end{aligned} \tag{4.77}$$

Now use (4.77) in place of (4.75) to obtain

$$\widetilde{\mathbf{L}}_m \le 1 + 2\,S_m(w_1, w_2),$$

or a similar estimate with $\overline{T}_m(w_1, w_2)$ if we assume $w_1 \in \mathbb{W}_{\rm qc}$. □

<u>Estimates for $\widetilde{\mathbf{L}}^*_m$ and $\mathbf{L}^*_m$:</u>

These can now be obtained applying the previous estimates to the system $\{\mathbf{e}^*_n, \mathbf{e}_n\}$, after interchanging the roles of w_1 and w_2 (and using the property $w_2 \in \mathbb{W}_{\rm qc}$ or $w_2 \in \mathbb{W}_{\rm co}$, respectively).

This completes the proof of all the asserted inequalities in Theorem 4.2.16, namely (4.67) and (4.68) since $w_1 = \varphi_u(m)$ and $w_2 = \varphi^*_u$ are quasi-concave. The optimality of the inequalities is illustrated in the Subsection 4.3.1 below.

4.2.5 Corollaries and Properties of $T_m(w_1, w_2)$

Some trivial corollaries that we can deduced using the main theorems of this section are the following.

Lemma 4.2.20. If $\mathbf{c} = \sup_n \|\mathbf{e}_n\|$ and $\mathbf{c}^* = \sup_n \|\mathbf{e}^*_n\|_*$, then

$$\overline{T}_m(\varphi_u, \varphi^*_u) \le \min\left\{\mathbf{c}^*\,\varphi_u(m), \mathbf{c}\,\varphi^*_u(m)\right\} \le \mathbf{c}\mathbf{c}^*\,m. \tag{4.78}$$

Proof. Using $\varphi_u(j) \le \mathbf{c}\,j$ (and $\varphi^*_u \in \mathbb{W}$), we deduce

$$T_m(\varphi_u, \varphi^*_u) \le \mathbf{c}\textstyle\sum_{j=1}^m \Delta\varphi^*_u(j) = \mathbf{c}\varphi^*_u(m).$$

Changing the roles of φ_u and φ^*_u the result follows easily. □

Remark 4.2.21. Inserting (4.78) in Theorem 4.2.16 one recovers the classical bound $\mathbf{L}_m \le 1 + 3\mathfrak{K}m$ (see Theorem 4.1.1), where $\mathfrak{K} := \sup_{n,m} \|\mathbf{e}_n\|\|\mathbf{e}^*_m\|_*$.

The next result could be applied quickly in some practical situations.

Corollary 4.2.22. Let $\{\mathbf{e}_n, \mathbf{e}^*_n\}_{n=1}^\infty$ be a complete semi-normalized biorthogonal system in $\mathbb{X}$. Then

(i) $\max\{\mathbf{L}_m, \mathbf{L}^*_m\} \lesssim \min\{\varphi_u(m), \varphi^*_u(m)\}$.

(ii) If $\min\{\varphi_u(m), \varphi^*_u(m)\} \lesssim k_m$, then $\mathbf{L}_m \approx k_m \approx \min\{\varphi_u(m), \varphi^*_u(m)\}$.

(iii) If $\varphi_l(m) \approx 1$, then $\mathbf{L}_m \approx \varphi_u(m)$.

Proof. (i) follows from Theorem 4.2.16 and the previous corollary.

(ii) follows from (i) and the known lower bound $\mathbf{L}_m \gtrsim k_m$ (Theorem 4.1.5).

(iii) Finally, if $\varphi_l(m) \approx 1$, then the super-democracy parameter takes the form $\tilde{\mu}_m \approx \varphi_u(m)$. So the result follows from (i) and the known lower bound $\mathbf{L}_m \gtrsim \mu_m$ (Theorem 4.1.5). $\square$

Now, we show some properties of the parameter $T_m(w_1,w_2)$ studying the relation between this one and other parameters that we have defined before. We show elementary relations for

$$S_m(w_1,w_2), \quad T_m(w_1,w_2), \quad \text{and} \quad \overline{T}_m(w_1,w_2),$$

defined in (4.65)-(4.66), and also for the quantity

$$U_m(w_1,w_2) := \sum_{j=1}^{m} \frac{w_1(j)\,w_2(j)}{j^2}, \quad m=1,2,\dots \tag{4.79}$$

Lemma 4.2.23. If $w_1,w_2 \in \mathbb{W}_{\rm qc}$ then

$$S_m(w_1,w_2) \le \overline{T}_m(w_1,w_2) \le \max\{T_m(w_1,w_2),T_m(w_2,w_1)\} \le U_m(w_1,w_2). \tag{4.80}$$

Moreover if we assume that $i_{w_1} i_{w_2} > 0$ then

$$\overline{T}_m(w_1,w_2) \approx U_m(w_1,w_2). \tag{4.81}$$

Finally, if $w_1 \in \mathbb{W}_{\rm co}$ and $i_{w_2} > 0$ (or $w_2 \in \mathbb{W}_{\rm co}$ and $i_{w_1} > 0$), then

$$S_m(w_1,w_2) \approx \overline{T}_m(w_1,w_2). \tag{4.82}$$

Proof. The assertion (4.80) follows easily using that $\Delta w(j) \le w(j)/j$ when $w \in \mathbb{W}_{\rm qc}$. If $i_{w_2} > 0$, we can apply Corollary 4.2.7 (ii) to obtain

$$U_m(w_1,w_2) = \|\{w_1(j)/j\}_{j=1}^m\|_{\ell^1_{w_2}} \le c\,\|\{w_1(j)/j\}_{j=1}^m\|_{\ell^1_{\widehat{w}_2}} = c\,T_m(w_1,w_2).$$

If $i_{w_1} > 0$, a symmetric argument gives $U_m(w_1,w_2) \le cT_m(w_2,w_1)$, and hence (4.81). Finally, if $w_1 \in \mathbb{W}_{\rm co}$ and $i_{w_2} > 0$, then Corollary 4.2.7 (ii) gives

$$T_m(w_2,w_1) = \|\{\Delta w_1(j)\}_{j=1}^m\|_{\ell^1_{w_2}} \le c\,\|\{\Delta w_1(j)\}_{j=1}^m\|_{\ell^1_{\widehat{w}_2}} = c\,S_m(w_1,w_2), \tag{4.83}$$

which together with (4.80) gives (4.82). A similar reasoning works interchanging w_1 and w_2. $\square$

Example 4.2.24. If $w_1(j) = j$ and $w_2(j) = 1$ for all $j = 1,2,3,\dots$. Then, $i_{w_1} = 1$, $i_{w_2} = 0$ and

$$S_m(w_1,w_2) = T_m(w_1,w_2) = 1, \quad T_m(w_2,w_1) = U_m(w_1,w_2) \approx \ln(m+1)\,.$$

Hence, (4.81) may not hold if $i_{w_2} = 0$.

4.3 Examples

4.3.1 The difference and summing bases

Let $(\mathbf{e}_n)_{n=1}^{\infty}$ denote the canonical basis in $\ell_1(\mathbb{N})$, and consider the system

$$\mathbf{x}_1 = \mathbf{e}_1\,, \quad \mathbf{x}_n = \mathbf{e}_n - \mathbf{e}_{n-1}\,,\ n = 2,3,\dots \tag{4.84}$$

This is a monotone basis in $\mathbb{X} = \ell_1$, sometimes called the **difference basis**. Observe that for finitely supported real scalars $\{b_n\}_{n=1}^{\infty}$ one has

$$\Big\|\sum_{n=1}^{\infty} b_n \mathbf{x}_n\Big\| = \sum_{n=1}^{\infty} |b_n - b_{n+1}|\,. \tag{4.85}$$

In particular, $\|\mathbf{x}_1\| = 1$ and $\|\mathbf{x}_n\| = 2$ if $n \geq 2$. The dual system consists of the ℓ_∞-vectors $\mathbf{x}_n^* = \sum_{m=n}^{\infty} \mathbf{e}_m^*$, so for $\{c_n\} \in c_{00}$ it holds that

$$\Big\|\sum_{n=1}^{\infty} c_n \mathbf{x}_n^*\Big\|_* = \sup_{n\geq 1}\Big|\sum_{j=1}^{n} c_j\Big|\,. \tag{4.86}$$

The system $(\mathbf{x}_n^*)_{n=1}^{\infty}$ is called the **summing basis** (see [63, p.20] and Example 4.1.2).

Lemma 4.3.1. For $\{\mathbf{x}_n, \mathbf{x}_n^*\}_{n=1}^{\infty}$ as above and $m = 1,2,3,\dots$, we have

(i) $\varphi_l(m) = 1$ and $\varphi_u(m) = 2m$

(ii) $\varphi_l^*(m) = 1$ and $\varphi_u^*(m) = m$.

Proof. For $A \subset \mathbb{N}, |A| = m$ and $\varepsilon \in \Upsilon_A$, if follows from (4.85) that

$$1 \leq \|\mathbf{1}_{\varepsilon A}\| = \Big\|\sum_{n\in A} \varepsilon_n \mathbf{x}_n\Big\| \leq 2m\,. \tag{4.87}$$

Using again (4.85), it is easily seen that the right equality in (4.87) is attained by testing with $\sum_{j=1}^{m} \mathbf{x}_{2j}$, while the left equality is attained with $\sum_{j=1}^{m} \mathbf{x}_j$. This shows the statements in (i) using that $\varphi_l(m) = h_l(m)$ since the basis is monotone. The statements in (ii) about the summing basis are similar (and can also be found in [18, Example 5.1]). □

Proposition 4.3.2. The system $\{\mathbf{x}_n, \mathbf{x}_n^*\}_{n=1}^{\infty}$ satisfies $S_m(\varphi_u, \varphi_u^*) = \overline{T}_m(\varphi_u, \varphi_u^*) = 2m$. Moreover,

$$k_m = k_m^* = 2m\,, \quad \widetilde{\mathbf{L}}_m = \widetilde{\mathbf{L}}_m^* = 1 + 4m\,, \quad \text{and} \quad \mathbf{L}_m = \mathbf{L}_m^* = 1 + 6m\,.$$

In particular, equalities are attained everywhere in Theorem 4.2.16.

Proof. From Lemma 4.3.1 we have

$$T_m(\varphi_u, \varphi_u^*) = \sum_{j=1}^{m} \frac{\varphi_u(j)}{j} \Delta\varphi_u^*(j) = 2m = T_m(\varphi_u^*, \varphi_u) = S_m(\varphi_u, \varphi_u^*)\,,$$

establishing the first assertion. Theorem 4.2.16 then implies

$$k_m^* \leq k_m \leq 2m, \quad \widetilde{\mathbf{L}}_m, \widetilde{\mathbf{L}}_m^* \leq 1+4m, \quad \text{and} \quad \mathbf{L}_m, \mathbf{L}_m^* \leq 1+6m.$$

The equalities for k_m^*, $\widetilde{\mathbf{L}}_m^*$ and $\mathbf{L}_m^*$ were shown in the Example 4.1.2. We show here that equalities are attained also for $\widetilde{\mathbf{L}}_m$ and $\mathbf{L}_m$. First consider

$$x = \sum_{j=1}^{2m+1} \mathbf{x}_j + \sum_{j=2m+1}^{3m} \mathbf{x}_{2j}.$$

Then, $\widetilde{\sigma}_m(x) \leq \left\| \sum_{j=1}^{2m+1} \mathbf{x}_j \right\| = 1$. However, choosing $G_m(x) = \sum_{j=1}^{m} \mathbf{x}_{2j}$ we have

$$\|x - G_m(x)\| = \left\| \sum_{j=1}^{m+1} \mathbf{x}_{2j-1} + \sum_{j=2m+1}^{3m} \mathbf{x}_{2j} \right\| = 4m+1.$$

Therefore, $\widetilde{\mathbf{L}}_m \geq \|x - G_m(x)\| / \widetilde{\sigma}_m(x) \geq 4m+1$. Finally, consider

$$x = \mathbf{x}_1 + \sum_{j=1}^{m} \mathbf{x}_{4j-2} + \sum_{j=1}^{m} \mathbf{x}_{4j-1} - \sum_{j=1}^{m} \mathbf{x}_{4j} + \sum_{j=1}^{m} \mathbf{x}_{4j+1}.$$

Taking $G_m(x) = \sum_{j=1}^{m} \mathbf{x}_{4j-2}$ we obtain $\|x - G_m(x)\| = 1+6m$. On the other hand, choosing $y = 2\sum_{j=1}^{m} \mathbf{x}_{4j} \in \Sigma_m$, we have

$$\sigma_m(x) \leq \|x+y\| = \left\| \sum_{j=1}^{4m+1} \mathbf{x}_j \right\| = 1.$$

Thus, $\mathbf{L}_m \geq \|x - G_m(x)\| / \sigma_m(x) \geq 1+6m$. □

4.3.2 The Lindenstrauss basis and its dual

We have introduced about this basis in Section 3.1 of Chapter 3. We remind some definitions. Let $(\mathbf{e}_n)_{n=1}^{\infty}$ denote the canonical basis in $\ell_1(\mathbb{N})$, and consider the vectors

$$\mathbf{x}_n = \mathbf{e}_n - \frac{1}{2}\mathbf{e}_{2n+1} - \frac{1}{2}\mathbf{e}_{2n+2}, \qquad n = 1,2,3,\dots \tag{4.88}$$

The system $\mathscr{L} = (\mathbf{x}_n)_{n=1}^{\infty}$ was introduced by J. Lindenstrauss in [62]. It is a basic sequence of ℓ_1, hence a basis of a subspace $\mathbb{D} = \overline{\operatorname{span}}\{\mathscr{L}\}$ in ℓ_1. To describe the dual system we consider the following vectors in c_0:

$$\mathbf{y}_n := \sum_{j=0}^{n} 2^{-j} \mathbf{e}_{\gamma_j(n)}, \quad n = 1,2,3,\dots \tag{4.89}$$

where $\gamma_0(n) = n$ and $\gamma_{j+1}(n) = \lfloor \frac{\gamma_j(n)-1}{2} \rfloor$, $j \geq 0$ (with the convention $\mathbf{e}_\gamma = 0$ if $\gamma \leq 0$). It is shown in [51, Example 2] that $\mathscr{Y} = (\mathbf{y}_n)_{n=1}^{\infty}$ is a Schauder basis in c_0 with dual vectors $\mathbf{y}_n^* = \mathbf{x}_n$.

In particular, there exists some $c>0$ such that

$$c\,\|y\|_{c_0} \le \sup_{\substack{x\in\mathbb{D}\\ \|x\|_{\ell_1}=1}} |\langle x,y\rangle| = \|y\|_{\mathbb{D}^*} \le \|y\|_{c_0}, \quad y\in c_0;$$

see e.g. [41, Exercise 6.12]. So we can identify $\mathbb{Y}=[\mathbf{y}_n : n\in\mathbb{N}]$ and c_0 with equivalent norms.

Some properties of $\{\mathscr{L},\mathscr{Y}\}$ are the following:

- $\mathscr{L}$ is conditional in $\mathbb{D}$, and $\mathbb{D}$ has no unconditional basis ([72, p. 454-457]).
- $\mathscr{L}$ is a quasi-greedy basis in $\mathbb{D}$, with $g_m\le 3$ for all m (see [39]).
- $\mathscr{Y}$ is not quasi-greedy in c_0 (see [39]).
- $k_m[\mathscr{L},\mathbb{D}]\approx \ln(m+1)\,, m=1,2,3,\ldots$ (see[4] [44, §6]).

Theorem 4.3.3. For the Lindenstrauss basis $\mathscr{L}$ in $\mathbb{D}$ we have $\overline{T}_m(\varphi_u,\varphi_u^*)\approx\ln(m+1)$. Moreover,

$$\widetilde{\mathbf{L}}_m\approx 1, \quad\text{and}\quad \mathbf{L}_m\approx\mathbf{L}_m^*\approx\widetilde{\mathbf{L}}_m^*\approx k_m\approx g_m^*\approx\mu_m^*\approx\ln(m+1). \tag{4.90}$$

Remark 4.3.4. The results for the system $\mathscr{Y}$ seem to be new. In fact, in this example, Theorem 4.2.16 performs better than Theorems 1.2 and 1.3 from [18], which would only yield the non-optimal bound $\mathbf{L}_m[\mathscr{Y},c_0]\lesssim[\ln(N+1)]^2$.

To prove the above result, we need the following lemma.

Lemma 4.3.5. For the Lindenstrauss basis $\mathscr{L}$ in $\mathbb{D}$ we have the following

(i) $\varphi_l(m)\approx m$ and $\varphi_u(m)=2m$.

(ii) $\varphi_l^*(m)\approx 1$ and $\varphi_u^*(m)\approx\ln(m+1)$.

Proof. i) Let $\mathbf{1}_{\varepsilon A}=\sum_{n\in A}\varepsilon_n\mathbf{x}_n$, with $|A|=m$, $\varepsilon\in\Psi_A$. Since $\|\mathbf{x}_n\|=2$, one always has $\|\mathbf{1}_{\varepsilon A}\|\le 2m$. To see that this bound is attained consider

$$\mathbf{x}=\sum_{n=1}^m \mathbf{x}_{3^n}=\sum_{n=1}^m\big(\mathbf{e}_{3^n}-\frac12\mathbf{e}_{2\cdot3^n+1}-\frac12\mathbf{e}_{2\cdot3^n+2}\big).$$

Since $2\cdot3^n+2<3^{n+1}$, one deduces that $\|\mathbf{x}\|=2m$. Hence $\varphi_u(m)=2m$.

We now give a lower estimate for $\varphi_l(m)$. Since the basis is Schauder, using Remark 1.2.3, $\varphi_l(m)\approx h_l(m)$. Observe that

$$\Big\|\sum_{n=1}^M b_n\mathbf{x}_n\Big\|_{\ell_1} = |b_1|+|b_2|+\sum_{n=3}^M\big|b_n-\tfrac12 b_{\lfloor\frac{n-1}{2}\rfloor}\big|+\tfrac12\sum_{n=M+1}^{2M+2}\big|b_{\lfloor\frac{n-1}{2}\rfloor}\big|.$$

From here it easily follows that $\|\mathbf{1}_{\varepsilon A}\|_{\ell_1}\ge|A|/2$, since for $n\in A$ we have $|b_n-\frac12 b_{\lfloor\frac{n-1}{2}\rfloor}|\ge1/2$. Thus

$$m/2\le h_l(m)\le 2m. \tag{4.91}$$

[4] This is shown in [44] for the system $\{\mathbf{e}_n-(\mathbf{e}_{2n}+\mathbf{e}_{2n+1})/2\}_{n=1}^\infty$, but the same arguments, with obvious modifications, work for the basis in (4.88).

ii) The inequality $\varphi_u^*(m) \gtrsim \ln(m+1)$ follows from

$$\Big\| \sum_{i=1}^{2^{m+1}-2} \mathbf{y}_i \Big\|_* \geq \frac{m}{2}\,; \tag{4.92}$$

see [39, (10)]. To give the upper bound, we proceed as follows: if $|A| = m$, as $\|\mathbf{1}^*_{\varepsilon A}\|_* = \sup_{\|x\|=1} |\mathbf{1}^*_{\varepsilon A}(x)|$, take $x \in \mathbb{X}$ with $\|x\| = 1$ and let ρ be the natural greedy ordering for x. If $\delta \equiv \{\mathrm{sign}(\mathbf{e}_j^*(x))\}$ and Λ_j is a greedy set of x of order j, using Proposition (2.1.12) with $C_q \leq 3$,

$$|\mathbf{e}_j^*(x)| \|\mathbf{1}_{\delta\Lambda_j}\| \leq 6. \tag{4.93}$$

Using (4.93) and the fact that $\varphi_l(m) \approx h_l(m) \approx m$,

$$\begin{aligned} |\mathbf{1}^*_{\varepsilon A}(x)| &= \Big| \sum_{n \in A} \varepsilon_n \mathbf{e}_n^*(x) \Big| \leq \sum_{j=1}^{m} |\mathbf{e}_j^*(x)| = \sum_{j=1}^{m} |\mathbf{e}_j^*(x)| \frac{\|\mathbf{1}_{\delta\Lambda_j}\|}{\|\mathbf{1}_{\delta\Lambda_j}\|} \\ &\leq 6 \sum_{j=1}^{m} \frac{1}{\varphi_l(m)} \lesssim \sum_{j=1}^{m} \frac{1}{m} \lesssim \ln(m+1). \end{aligned}$$

To estimate $\varphi_l^*(m)$ we quote the equality (9) in [39],

$$\Big\| \sum_{i=1}^{2^{m+1}-2} (-1)^i \mathbf{y}_i \Big\|_{c_0} = 1\,. \tag{4.94}$$

Since $\mathscr{Y}$ is a Schauder basis, this actually implies that $h_l^*(m) \lesssim 1$. On the other hand, given any $A \subset \mathbb{N}$, if we set $n_0 = \min A$, then

$$\| \sum_{n \in A} \varepsilon_n \mathbf{y}_n \|_{c_0} \geq \frac{1}{\mathfrak{K}_b} \|\mathbf{y}_{n_0}\|_{c_0} = 1,$$

which implies $h_l^*(m) \gtrsim 1$. Since $\mathscr{Y}$ is Schauder, $\varphi_l^* \approx h_l^* \approx 1$. □

Proof of Theorem 4.3.3. By Lemma 4.3.5 we have $\varphi_u(j) = 2j$, and therefore,

$$S_m(\varphi_u, \varphi_u^*) = T_m(\varphi_u, \varphi_u^*) = 2\varphi_u^*(m) \approx \ln(m+1). \tag{4.95}$$

Thus, Theorem 4.2.16 gives a logarithmic upper bound for all the quantities in (4.90). Also, $\widetilde{\mathbf{L}}_m \approx 1$ is known from [39] (since $\mathscr{L}$ is quasi-greedy and democratic).

For the lower bounds, first note that $\mathbf{L}_m \gtrsim k_m \gtrsim \ln(m+1)$ was shown in [44, §6.1]. Lemma 4.3.5 also gives $\mu_m^* \approx \ln(m+1)$. Finally, $\mathbf{L}_m^* \geq \widetilde{\mathbf{L}}_m^* \gtrsim g_m^*$, and the estimate $g_m^* \gtrsim \ln(m+1)$ can easily be obtained from (4.92) and (4.94). □

4.3.3 The trigonometric system in $L_p(\mathbb{T})$

Consider $\mathscr{B} = (e^{inx})_{n\in\mathbb{Z}}$ in $L_p(\mathbb{T})$ for $1 \le p < \infty$, and in $C(\mathbb{T})$ if $p = \infty$ (we equip $\mathscr{B}$ with its natural ordering $\{1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, \dots\}$). In [76], Temlyakov showed that

$$c_p m^{|\frac{1}{p}-\frac{1}{2}|} \le \mathbf{L}_m \le 1 + 3m^{|\frac{1}{p}-\frac{1}{2}|},$$

for some $c_p > 0$ and all $1 \le p \le \infty$.

Also, he showed that $g_m \approx k_m \approx \mathbf{L}_m \approx m^{|\frac{1}{p}-\frac{1}{2}|}$ for any $1 \le p \le \infty$. The tools that he used to prove this result are based on some ideas of functional analysis and harmonic analysis. Here, we present the same bound using our Theorem 4.2.16.

Proposition 4.3.6. Let $\mathscr{B} = (e^{inx})_{n\in\mathbb{Z}}$ in $L_p(\mathbb{T})$ for $1 \le p < \infty$, and in $C(\mathbb{T})$ if $p = \infty$. For $p \ne 2$,

$$\overline{T}_m(\varphi_u, \varphi_u^*) \approx \mathbf{L}_m \approx \widetilde{\mathbf{L}}_m \approx k_m \approx \mathbf{L}_m^* \approx \widetilde{\mathbf{L}}_m^* \approx m^{|\frac{1}{p}-\frac{1}{2}|}. \tag{4.96}$$

Proof. From the Hausdorff-Young inequality and elementary inclusions, it is straightforward to prove that

$$m^{\frac{1}{2}\wedge\frac{1}{p'}} \le \|\mathbf{1}_{\varepsilon A}\|_p \le m^{\frac{1}{2}\vee\frac{1}{p'}}, \tag{4.97}$$

for all $A \in \mathbb{N}^m$ and $\varepsilon \in \Psi_A$. Thus,

$$\varphi_u(m) \le m^{\frac{1}{2}\vee\frac{1}{p'}} \quad \text{and} \quad \varphi_u^*(m) \le m^{\frac{1}{2}\vee\frac{1}{p}},$$

and therefore

$$\overline{T}_m(\varphi_u, \varphi_u^*) \le U_m(\varphi_u, \varphi_u^*) = \sum_{j=1}^{m} \frac{j^{|\frac{1}{p}-\frac{1}{2}|}}{j} \le c_p m^{|\frac{1}{p}-\frac{1}{2}|},$$

with $c_p = 1/|\frac{1}{p} - \frac{1}{2}|$. This and Theorem 4.2.16 provide upper bounds for the constants in (4.96). The lower bounds follow from $g_m \gtrsim m^{|\frac{1}{p}-\frac{1}{2}|}$ (see [76, Remark 2]). $\square$

Remark 4.3.7. When $\mathbb{X} = L_2$ one of course has $k_m = \widetilde{\mathbf{L}}_m = \mathbf{L}_m = 1$. Observe, however, that $\varphi_u(j) = \varphi_u^*(j) = \sqrt{j}$ only gives $\overline{T}_m \approx \ln(m+1)$. This loss is due to the fact that, in Theorem 4.2.16, we only make use of the weak assumptions $\ell_{2,1} \hookrightarrow \mathbb{X} \hookrightarrow \ell_{2,\infty}$, rather than the full force of $\mathbb{X} = \ell_2$.

For the parameters μ_m and $\tilde{\mu}_m$ for $1 \le p \le \infty$, we have the following bounds.

- First, consider the case $p = 1$. Assume that $m = 2\ell + 1$ or $2\ell + 2$ (that is, $\ell = \lfloor \frac{m-1}{2} \rfloor$). Choose $B = \{-\ell, ..., \ell\}$, so that $\mathbf{1}_B = D_\ell$ is the ℓ-th Dirichlet kernel, and hence

  $$\|\mathbf{1}_B\|_1 = \|D_\ell\|_1 \approx \ln(m+1).$$

  Hence,

  $$\tilde{\mu}_{m+1} \ge \mu_{m+1} \ge \frac{\|\mathbf{1}_{\{1,2,\dots,2^m\}}\|_1}{\|\mathbf{1}_B\|_1} \gtrsim \frac{\sqrt{m}}{\ln(m+1)},$$

  where $\|\mathbf{1}_{\{1,2,\dots,2^m\}}\|_1 \gtrsim \sqrt{m}$ by [57, pg. 121].

The upper bound relies on $\|\mathbf{1}_{\varepsilon' B}\|_1 \le \|\mathbf{1}_{\varepsilon' B}\|_2 = |B|^{\frac{1}{2}}$ for any finite set B and $\varepsilon' \in \Psi_B$, and on the deeper result

$$\inf_{\varepsilon,|A|=m} \|\mathbf{1}_{\varepsilon A}\|_1 \ge c\ln(m+1),$$

a famous problem posed by Littlewood and solved by Konyagin [59] and McGeehee-Pigno-Smith [64]. Hence, for any $k \ge m$, $\varphi_l(k) \ge \ln(m+1)$. Thus,

$$\tilde{\mu}_m \approx \mu_m \approx \frac{\sqrt{m}}{\ln(m+1)}.$$

- If $1 < p \le 2$ and m is even,

$$\mu_{m+1} \ge \frac{\|\mathbf{1}_{\{1,2,\dots,2^m\}}\|_p}{\|D_\ell\|_p} \gtrsim \frac{\sqrt{m}}{m^{1-1/p}} = m^{h(p)},$$

where $h(p) = \frac{1}{p} - \frac{1}{2}$, D_ℓ is the Dirichlet kernel defined in the above case and it is well known that $\|D_\ell\|_p \approx m^{1-1/p}$ when $1 < p < \infty$. Moreover, since $\mu_m \le \tilde{\mu}_m \le \mathbf{L}_m \lesssim m^{h(p)}$,

$$\mu_m \approx \tilde{\mu}_m \approx m^{h(p)}.$$

- If $2 \le p < \infty$, we only have to replace the roles of numerator and denominator.

- For the case $p = \infty$, it is clear that the trigonometric system is democratic. To show that $\tilde{\mu}_m \approx \sqrt{m}$, we use the Rudin-Shapiroo polynomials:

$$R(x) = e^{iNx} \sum_{n=0}^{2^L-1} \varepsilon_n e^{inx}, \quad \text{with } \varepsilon_n \in \{\pm 1\},$$

where L is such that $2^L \le m < 2^{L+1}$ (see e.g. [57, p. 33]). Then, $R = \mathbf{1}_{\varepsilon B}$ with $B = N + \{0,1,\dots,2^L-1\}$ and

$$\|\mathbf{1}_{\varepsilon B}\|_\infty = \|R\|_\infty \approx \sqrt{m}.$$

If we pick $N \ge 2 \cdot 2^L$, then $B > 2A$ with $A = \{\pm 1, \dots, \pm(2^L-1)\}$. Finally,

$$\|\mathbf{1}_A\|_\infty = \|D_{2^L-1} - 1\|_\infty \approx m.$$

Hence, $\tilde{\mu}_m \gtrsim \sqrt{m}$. Now, since $\tilde{\mu}_m \lesssim \mathbf{L}_m \lesssim \sqrt{m}$, $\tilde{\mu}_m \approx \sqrt{m}$.

Remark 4.3.8. Since all of the parameters μ_m, $\tilde{\mu}_m$, g_m and k_m are of the same order, Theorems 4.1.3 and 4.1.5 are not optimal to find the bound of $\mathbf{L}_m$.

4.3.4 A summing basis by blocks.

This is a slight modification of an example exhibited in [44, Proposition 7.1]. It again illustrates that Theorem 4.2.16 produces asymptotically optimal bounds, which cannot be obtained with the results in [18]. Take any $\{\omega_j\}_{j=1}^\infty \in \mathbb{W}_{\text{qc}}$, say with $\omega_1 = 1$. Define a space $\mathbb{X}$ consisting of (real) sequences $x = (x_n)_{n=1}^{\infty} \in c_0$ such that

$$\|x\| = \max\left\{ \|x\|_\infty,\ \sup_{j\geq 1}\sup_{N\geq 1} \frac{\omega_j}{j}\Big|\sum_{\substack{n\in\Delta_j\\ n\leq N}} x_n\Big| \right\} < \infty,$$

where $\Delta_j = \{2^j, ..., 2^j + 2j - 1\}$, $j = 1, 2, ...$ By definition of the norm, the canonical system $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty}$ verifies $\|\mathbf{e}_n\| = \|\mathbf{e}_n^*\|_* = 1$ for all n.

Proposition 4.3.9. In this example we have $\overline{T}_m(\varphi_u, \varphi_u^*) \leq 2\omega_m$, and therefore

$$k_m \leq 2\omega_m, \quad \widetilde{\mathbf{L}}_m \leq 1 + 4\omega_m, \quad \text{and} \quad \mathbf{L}_m \leq 1 + 6\omega_m, \qquad m = 1, 2, \ldots \tag{4.98}$$

Moreover, all these quantities are bounded below by $\min\{g_m, g_m^c\} \geq \omega_m$.

Proof. For any $A \in \mathbb{N}^m$ and $\varepsilon \in \Psi_A$ we claim that

$$1 \leq \|\mathbf{1}_{\varepsilon A}\| \leq \|\mathbf{1}_A\| = \max\left\{1, \sup_j \frac{\omega_j}{j}|\Delta_j \cap A|\right\} \leq 2\omega_m. \tag{4.99}$$

Indeed, the last inequality is justified using the quasi-concavity of ω as follows:

- if $j \geq m$, then $\frac{\omega_j}{j}|\Delta_j \cap A| \leq \frac{\omega_j}{j}|A| = \frac{\omega_j}{j} m \leq \omega_m$
- if $j \leq m$, then $\frac{\omega_j}{j}|\Delta_j \cap A| \leq \frac{\omega_j}{j}|\Delta_j| = 2\omega_j \leq 2\omega_m$.

On the other hand, we have the trivial estimate $\|\mathbf{1}^*_{\varepsilon A}\|_* \leq |A|$. Therefore, arguing as in Corollary 4.2.20 we obtain $\overline{T}_m(\varphi_u, \varphi_u^*) \leq 2\omega_m$, and therefore (4.98) holds due to Theorem 4.2.16. We now show the lower bound. Let $x = \sum_{j=0}^{2m-1}(-1)^j \mathbf{e}_{2^m+j}$, which has support in Δ_m and $\|x\| = 1$. Choosing $G_m(x) = \sum_{\ell=0}^{m-1} \mathbf{e}_{2^m+2\ell}$, we see that

$$g_m \geq \|G_m(x)\| = \omega_m \quad \text{and} \quad g_m^c \geq \|(\mathrm{Id}_{\mathbb{X}} - G_m)(x)\| = \omega_m.$$

□

4.3.5 An example of Konyagin and Temlyakov

We slightly generalize a construction in [60] of a quasi-greedy superdemocratic basis which is not unconditional. For $1 \leq p < \infty$ and $1 \leq r \leq \infty$, let $KT(p,r)$ be the set of all sequences $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in c_0$ with norm

$$|||\mathbf{x}||| = \max\left\{\|\mathbf{x}\|_{\ell_{p,r}},\ \|\mathbf{x}\|_{b_p}\right\} < \infty$$

where

$$\|\mathbf{x}\|_{\ell_{p,r}} = \left(\sum_{j=1}^{\infty} (j^{1/p}x_j^*)^r \frac{1}{j}\right)^{1/r}, \quad \text{and} \quad \|\mathbf{x}\|_{b_p} = \sup_{k\geq 1}\Big|\sum_{n=1}^{k} \frac{x_n}{n^{1/p'}}\Big|.$$

The example in [60, §3.3] is the case $KT(2,2)$, while $KT(p,p), 1 < p < \infty$, was later considered in [44]. A trivial case corresponds to $r = 1$, for which $K(p,1) = \ell_{p,1}$.

We summarize the main results in the next theorem, where we write $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ for the standard canonical basis, that is Schauder for every $1 \le p < \infty$ and $1 \le r \le \infty$ in $KT(p,r)$.

Theorem 4.3.10. Let $1 \le r \le \infty$.

(i) If $1 < p < \infty$ then $\big(KT(p,r),\mathscr{B}\big)$ is quasi-greedy, bidemocratic and

$$\mathbf{L}_m \approx \mathbf{L}_m^* \approx k_m \approx [\ln(m+1)]^{1/r'} \quad \text{and} \quad \widetilde{\mathbf{L}}_m \approx \widetilde{\mathbf{L}}_m^* \approx 1. \tag{4.100}$$

(ii) If $p = 1$ then $\big(KT(1,r),\mathscr{B}\big)$ is super-democratic and

$$\overline{T}_m(\varphi_u,\varphi_u^*) \approx \mathbf{L}_m \approx \widetilde{\mathbf{L}}_m \approx \mathbf{L}_m^* \approx \widetilde{\mathbf{L}}_m^* \approx k_m \approx g_m \approx \mu_m^* \approx [\ln(m+1)]^{1/r'}. \tag{4.101}$$

We split the proof in various lemmas, starting with the computation of φ_u and φ_u^*.

Lemma 4.3.11. If $1 \le r \le \infty$, the following holds for the space $KT(p,r)$.

(i) If $1 < p < \infty$, then $h_l(m) \approx \varphi_u(m) \approx m^{1/p}$, and $\varphi_l^*(m) \approx \varphi_u^*(m) \approx m^{1/p'}$.

(ii) If $p = 1$, then $\varphi_l(m) \approx \varphi_u(m) \approx m$, $\varphi_l^*(m) \approx 1$ and $\varphi_u^*(m) \approx [\ln(m+1)]^{1/r'}$.

In particular, $\big(KT(p,r),\mathscr{B}\big)$ is always super-democratic, and is bidemocratic if $p > 1$.

Proof. If $A \in \mathbb{N}^m$ and $\varepsilon \in \Psi_A$, then

$$|||\mathbf{1}_{\varepsilon A}||| \le |||\mathbf{1}_A||| \le \max\Big\{[\sum_{j=1}^m (j^{\frac{1}{p}})^r \tfrac{1}{j}]^{\frac{1}{r}}, \sum_{j=1}^m \tfrac{1}{j^{1/p'}}\Big\} = \sum_{j=1}^m \tfrac{1}{j^{1/p'}} \le p m^{1/p}, \tag{4.102}$$

and

$$|||\mathbf{1}_{\varepsilon A}||| \ge \|\mathbf{1}_{\varepsilon A}\|_{\ell_{p,r}} = [\sum_{j=1}^m (j^{\frac{1}{p}})^r \tfrac{1}{j}]^{\frac{1}{r}} \ge c_{p,r} m^{1/p},$$

for some $c_{p,r} > 0$. This shows that $h_l(m) \approx \varphi_u(m) \approx m^{1/p}$ for all $1 \le p < \infty$. Moreover, since the basis is Schauder, $\varphi_l(m) \approx h_l(m)$, hence, the basis is super-democratic. For the assertion about the dual system, observe that if $|||\mathbf{x}||| = 1$, then

$$\begin{aligned} |\mathbf{1}_{\varepsilon A}^*(\mathbf{x})| &\le \sum_{n\in A} |x_n| \le \sum_{j=1}^m x_j^* \\ &\le \|\mathbf{x}\|_{\ell^{p,r}} \Big[\sum_{j=1}^m j^{\frac{r'}{p'}} \tfrac{1}{j}\Big]^{\frac{1}{r'}} \le \begin{cases} m^{1/p'} & \text{if } 1 < p < \infty \\ [\ln(m+1)]^{\frac{1}{r'}} & \text{if } p = 1 \end{cases} \end{aligned} \tag{4.103}$$

So taking sup over $|||\mathbf{x}||| = 1$ we obtain the asserted upper bounds for $\varphi_u^*(m)$. For the lower bound, using (4.102),

$$|||\mathbf{1}_{\varepsilon A}^*|||_* \ge \mathbf{1}_{\varepsilon A}^*(\mathbf{1}_{\bar{\varepsilon} A}) / |||\mathbf{1}_{\bar{\varepsilon} A}||| \ge m/(p m^{\frac{1}{p}}) = m^{\frac{1}{p'}}/p. \tag{4.104}$$

So, when $1<p<\infty$ we have already proved $\varphi_l^*(m)\approx\varphi_u^*(m)\approx m^{1/p'}$. When $p=1$, one can obtain $\varphi_l^*(m)\approx 1$ from (4.104),

$$|||\mathbf{1}^*_{\{1,\dots,m\}}|||_* = \sup_{|||\mathbf{x}|||=1}\Big|\sum_{n=1}^m x_n\Big|\le 1,$$

and the fact that $h_l^*(m)\approx\varphi_l^*(m)$.

Finally, setting $\varepsilon_n=(-1)^n$ and $\mathbf{x}=\sum_{n=1}^m\frac{(-1)^n}{n}\mathbf{e}_n$, we have $|||\mathbf{x}|||\approx[\ln(m+1)]^{1/r}$ and therefore

$$|||\mathbf{1}^*_{\varepsilon\{1,\dots,m\}}|||_*\ge\big|\textstyle\sum_{n=1}^m\frac{1}{n}\big|/|||\mathbf{x}|||\approx[\ln(m+1)]^{1/r'}.$$

This and (4.103) show that $\varphi_u^*(m)\approx[\ln(m+1)]^{1/r'}$, and establish the lemma. □

The following proof is a variation of [60, §3.4].

Lemma 4.3.12. Let $1<p<\infty$ and $1\le r\le\infty$. Then $\mathscr{B}$ is quasi greedy in $KT(p,r)$.

Proof. Since the canonical basis is unconditional in $\ell_{p,r}$ and $KT(p,1)=\ell^{p,1}$ we may assume that $r>1$. Also, it suffices to show that $\|G_m(\mathbf{x})\|_{b_p}\le C|||\mathbf{x}|||$, for all $G_m\in\mathscr{G}_m$ and all m. Let $\mathbf{x}\in KT(p,r)$, $\Lambda\in\mathscr{G}(\mathbf{x},m)$, $\alpha=\min_{j\in\Lambda}x_j^*$ and $M_\alpha=\left(\frac{|||\mathbf{x}|||}{\alpha}\right)^p\ge 1$.

Then, for $M\le M_\alpha$, using that $|x_j|\le\alpha$ if $j\in\Lambda^c$, we obtain

$$\begin{aligned}\Bigg|\sum_{\substack{j=1\\ j\in\Lambda}}^M\frac{x_j}{j^{1/p'}}\Bigg| &\le \Bigg|\sum_{j=1}^M\frac{x_j}{j^{1/p'}}\Bigg|+\Bigg|\sum_{\substack{j=1\\ j\in\Lambda^c}}^M\frac{x_j}{j^{1/p'}}\Bigg|\le\|\mathbf{x}\|_{b_p}+\alpha\sum_{j=1}^{M_\alpha}\frac{1}{j^{1/p'}}\\ &\lesssim |||\mathbf{x}|||+\alpha M_\alpha^{1/p}\lesssim|||\mathbf{x}|||.\end{aligned}\tag{4.105}$$

For $M>M_\alpha$, we use (4.105) to obtain

$$\Bigg|\sum_{\substack{j=1\\ j\in\Lambda}}^M\frac{x_j}{j^{1/p'}}\Bigg|\le\Bigg|\sum_{\substack{j=1\\ j\in\Lambda}}^{M_\alpha}\frac{x_j}{j^{1/p'}}\Bigg|+\Bigg|\sum_{\substack{M_\alpha<j\le M\\ j\in\Lambda}}\frac{x_j}{j^{1/p'}}\Bigg|\lesssim|||\mathbf{x}|||+\underbrace{\Bigg|\sum_{\substack{M_\alpha<j\le M\\ j\in\Lambda}}\frac{x_j}{j^{1/p'}}\Bigg|}_{(I)}.\tag{4.106}$$

To estimate (I), take a number q such that $\max\{1,p/r\}<q<p$. Set $s=rq/p>1$ (if $r=\infty$, then $s=\infty$ as well). By the Hardy-Littlewood rearragement inequality,

$$\begin{aligned}(I) &\le \sum_{j=1}^m\frac{x_j^*}{(j+M_\alpha)^{1/p'}}\le\alpha^{1-p/q}\sum_{j=1}^m\frac{(x_j^*)^{p/q}j^{1/q}j^{1/q'}}{(j+M_\alpha)^{1/p'}}\frac{1}{j}\\ &\le \alpha^{1-p/q}\left(\sum_{j=1}^\infty(j^{1/p}x_j^*)^{sp/q}\frac{1}{j}\right)^{1/s}\left(\sum_{j=1}^\infty\left(\frac{j^{1/q'}}{(j+M_\alpha)^{1/p'}}\right)^{s'}\frac{1}{j}\right)^{1/s'}\\ &\le \alpha^{1-p/q}|||\mathbf{x}|||^{p/q}\underbrace{\left(\sum_{j=1}^\infty\frac{j^{s'/q'}}{(j+M_\alpha)^{s'/p'}}\frac{1}{j}\right)^{1/s'}}_{(II)}.\end{aligned}$$

Finally, we estimate (II) as follows:

$$\begin{aligned}(II) &\leq M_\alpha^{-1/p'}\left(\sum_{j\leq M_\alpha}\frac{j^{s'/q'}}{j}\right)^{1/s'}+\left(\sum_{j>M_\alpha}\frac{1}{j^{(\frac{1}{p'}-\frac{1}{q'})s'}}\frac{1}{j}\right)^{1/s'}\\ &\lesssim M_\alpha^{1/q'-1/p'}\leq\left(\|\|\mathbf{x}\|\|/\alpha\right)^{p(1/p-1/q)}. \qquad (4.107)\end{aligned}$$

Hence, using (4.107) in the estimate of (I),

$$(I)\lesssim\alpha^{1-p/q}\|\|\mathbf{x}\|\|^{p/q}\|\|\mathbf{x}\|\|^{1-p/q}/\alpha^{1-p/q}=\|\|\mathbf{x}\|\|. \qquad (4.108)$$

Thus (4.108), (4.106), and (4.105) show that $\|G_m(\mathbf{x})\|_{b_p}\lesssim\|\|\mathbf{x}\|\|$, establishing the result. □

Lemma 4.3.13. For $1\leq p<\infty$ and $1\leq r\leq\infty$, we have $k_m\gtrsim(\ln(m+1))^{1/r'}$. In particular, $\mathscr{B}$ is not unconditional in $KT(p,r)$ if $r>1$.

Proof. Consider $\mathbf{x}=\sum_{n=1}^{2m}\frac{(-1)^n}{n^{1/p}}\mathbf{e}_n$, with $m\geq 1$. Then,

$$\|\|\mathbf{x}\|\|=\left(\sum_{n=1}^{2m}\tfrac{1}{n}\right)^{1/r}\approx[\ln(m+1)]^{1/r}.$$

On the other hand, for the set $A=\{1,2,...,2m\}\cap 2\mathbb{Z}$, with cardinality m, then,

$$\|\|P_A(\mathbf{x})\|\|\geq\|P_A(\mathbf{x})\|_{b_p}=\sum_{n=1}^{m}\tfrac{1}{2n}\approx\ln(m+1).$$

Thus, $k_m\geq\|\|P_A(\mathbf{x})\|\|/\|\|\mathbf{x}\|\|\gtrsim[\ln(m+1)]^{1/r'}$. □

Lemma 4.3.14. For all $1\leq r\leq\infty$, the space $KT(1,r)$ satisfies $g_m\gtrsim(\ln(m+1))^{1/r'}$. In particular, $\mathscr{B}$ is not quasi-greedy in $KT(1,r)$ if $r>1$.

Proof. For fixed $n\geq 1$, consider

$$\mathbf{x}=\Big(1,\underbrace{-\frac{1}{2^n},...,-\frac{1}{2^n}}_{2^n\text{elements}},\frac{1}{2},\frac{1}{2},\underbrace{-\frac{1}{2^{n+1}},...,-\frac{1}{2^{n+1}}}_{2^{n+1}\text{elements}},...,\frac{1}{2^n},...,\frac{1}{2^n},\underbrace{-\frac{1}{2^{2n}},...,-\frac{1}{2^{2n}}}_{2^{2n}\text{elements}},0,\dots\Big).$$

Then $\|\mathbf{x}\|_{b_1}=1$, and since the decreasing rearrangement of $\mathbf{x}$ is given by

$$\Big(1,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},...,\frac{1}{2^{2n}},...,\frac{1}{2^{2n}},0,\dots\Big),$$

we also have $\|\mathbf{x}\|_{\ell^{1,r}}\approx[\sum_{j=0}^{2n}(2^j x_{2^j}^*)^r]^{1/r}=[2n+1]^{1/r}\approx\|\|\mathbf{x}\|\|$.

Now, if $m=1+2+...+2^n=2^{n+1}-1$, then

$$G_m(\mathbf{x})=\Big(1,0,...0,\frac{1}{2},\frac{1}{2},0,...,0,...,\frac{1}{2^n},...,\frac{1}{2^n}\Big),$$

and therefore $\|G_m(\mathbf{x})\|_{b_1}=n+1$. Hence, $\|\|G_m(\mathbf{x})\|\|\geq n+1=\log_2(m+1)$, and we conclude

$$g_m\geq\|\|G_m(\mathbf{x})\|\|/\|\|\mathbf{x}\|\|\gtrsim(n+1)^{1/r'}\approx[\ln(m+1)]^{1/r'}.$$

□

Proof of Theorem 4.3.10. Assume first that $1<p<\infty$. From Lemmas 4.3.11 and 4.3.12, $\mathscr{B}$ is quasi-greedy and bidemocratic, so the dual basis $\mathscr{B}^*$ is also quasi-greedy ([32, Theorem 5.4]). Thus, by [32, Theorem 3.3], $\widetilde{\mathbf{L}}_m \approx \widetilde{\mathbf{L}}_m^* \approx 1$, as asserted in (4.100). Also $\mathbf{L}_m \approx \mathbf{L}_m^* \approx k_m$, by [44, Theorem 1.1], and hence the lower bounds on the left side of (4.100) follow from Lemma 4.3.13. We must give an upper bound for k_m. We shall use a direct argument, based on the fact that $KT(p,r) \hookrightarrow \ell_{p,r}$. Going back to (4.70) in the proof of Theorem 5.3.9, first notice that we can choose the sequence $w_1(m)=\sum_{j=1}^m 1/j^{1/p'}$ because of (4.102). Then, for $|A|\le m$,

$$\begin{aligned}
|||P_A(\mathbf{x})||| &\le \sum_{j=1}^m a_j^*(x)\Delta w_1(j) = \sum_{j=1}^m x_j^* \frac{j^{1/p}}{j} \\
&\le \Big(\sum_{j=1}^m (x_j^* j^{1/p})^r \frac{1}{j}\Big)^{1/r} \Big(\sum_{j=1}^m \frac{1}{j}\Big)^{1/r'} \le |||\mathbf{x}|||\,[\ln(m+1)]^{1/r'}.
\end{aligned}$$

This gives a direct bound $k_m \le [\ln(m+1)]^{1/r'}$, and completes the proof of the theorem for the case $p>1$.

Assume now that $p=1$. Since $\min\{\widetilde{\mathbf{L}}_m, \mathbf{L}_m, k_m\} \gtrsim g_m$ and $\min\{\widetilde{\mathbf{L}}_m^*, \mathbf{L}_m^*\} \ge \mu_m^*$, the lower bounds follow from Lemmas 4.3.11 and 4.3.14. To establish the upper bounds, we shall give a direct argument that avoids Theorems 4.2.16, 4.2.9 and 4.2.14, as $\mathbb{X}=KT(1,r)$ is only a quasi-Banach space. As before, the trivial embedding $\ell_1 \hookrightarrow KT(1,r)$ gives

$$\begin{aligned}
|||P_A(\mathbf{x})||| &\lesssim \|P_A(\mathbf{x})\|_{\ell_1} = \sum_{j=1}^{|A|} x_j^* \\
&\le \left(\sum_{j=1}^{|A|} (jx_j^*)^r \frac{1}{j}\right)^{1/r} \left(\sum_{j=1}^{|A|} \frac{1}{j}\right)^{1/r'} \\
&\lesssim |||\mathbf{x}|||\,(\ln(|A|+1))^{1/r'},
\end{aligned} \tag{4.109}$$

from which one derives $k_m \lesssim (\ln(m+1))^{1/r'}$. To obtain an upper bound for $\mathbf{L}_m$, using the notation and the arguments following (4.74), one has

$$II = |||P_{B\setminus\Gamma}(x)||| \lesssim \big\|P_{B\setminus\Gamma}(x)\big\|_{\ell_1} = \sum_{j=1}^{|B\setminus\Gamma|} a_j^*\big(P_{B\setminus\Gamma}(x)\big) \le \sum_{j=1}^{|\Gamma\setminus B|} a_j^*(x-z).$$

So using again (4.109) one obtains

$$II \lesssim |||x-z|||\,\big[\ln(N+1)\big]^{\frac{1}{r'}}.$$

From these expressions, the arguments in (4.74) and (4.75) lead to

$$|||x-G_m(x)||| \lesssim |||x-z|||\,\big[\ln(N+1)\big]^{\frac{1}{r'}},$$

and hence $\mathbf{L}_m \lesssim (\ln(m+1))^{1/r'}$. Finally, a bound for $\mathbf{L}_m^*$ can be obtained similarly as follows. If $x \in \mathbb{X}^*$, we use the expression for the dual norm

$$II = \||P_{B\setminus\Gamma}(x)\||_* = \sup_{\||y\||=1} \left|\langle P_{B\setminus\Gamma}(x), y\rangle\right|$$

Now, for fixed $\||y\|| = 1$, the Hardy-Littlewood rearrangement inequality ([24, Theorem 2.2, Chapter 2]) and the reasoning following (4.75) give

$$\begin{aligned} \left|\langle P_{B\setminus\Gamma}(x), y\rangle\right| &\leq \sum_{j=1}^{|B\setminus\Gamma|} a_j^*\big(P_{B\setminus\Gamma}(x)\big) y_j^* \leq \sum_{j=1}^{|\Gamma\setminus B|} a_j^*(x-z) y_j^* \\ &\leq \Big[\sum_{j=1}^{\infty} (j y_j^*)^r \tfrac{1}{j}\Big]^{\frac{1}{r}} \Big(\sum_{j=1}^{|\Gamma\setminus B|} a_j^*(x-z)^{r'} \tfrac{1}{j}\Big)^{\frac{1}{r'}} \\ &\lesssim \||y\|| \, \|x-z\|_{\ell_\infty} \big[\ln(m+1)\big]^{\frac{1}{r'}}. \end{aligned}$$

Since $KT(1,r)^* \hookrightarrow \ell_\infty$ we obtain

$$II \lesssim \||x-z\||_* \big[\ln(m+1)\big]^{\frac{1}{r'}},$$

and conclude that

$$\||x - G_m(x)\||_* \lesssim \||x-z\||_* \big[\ln(m+1)\big]^{\frac{1}{r'}}.$$

Thus, we have also shown $\mathbf{L}_m^* \lesssim (\ln(m+1))^{1/r'}$, and completed the proof of Theorem 4.3.10.

□

4.4 Bidemocracy and Lebesgue-type parameters

In this section we present an alternative upper bound of $\mathbf{L}_m$ in terms of the unconditional and bidemocratic parameter. For that, we define the last one:

$$\mathbf{B}_m[\mathscr{B},\mathbb{X}] := \mathbf{B}_m = \sup_{r\leq m} \frac{\varphi_u[\mathscr{B},\mathbb{X}](r)\, \varphi_u^*[\mathscr{B}^*,\mathbb{Y}](r)}{r},$$

where $\mathbb{Y} = [\mathscr{B}^*]$. We will use $\varphi_u(r)$ and $\varphi_u^*(r)$ to denote $\varphi_u[\mathscr{B},\mathbb{X}](r)$ and $\varphi_u^*[\mathscr{B}^*,\mathbb{Y}](r)$, respectively. Following the spirit of Proposition 3.1.10, we prove the following results.

Lemma 4.4.1. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. For every $x \in \mathbb{X}$ and $m \in \mathbb{N}$,

$$a_m^*(x)\, \varphi_u(m) \leq \mathbf{B}_m \|x\|,$$

where $a_m^*(x) = |\mathbf{e}_{\pi(m)}^*(x)|$ for π a greedy ordering.

Proof. Let $G \subseteq \operatorname{supp}(x)$ be such that $|G| = m$ and $a_m^*(x) \le |\mathbf{e}_j^*(x)|$ for all $j \in G$. Define $f^* \in \mathbb{X}^*$ by $f^* = \sum_{j\in G} \overline{\operatorname{sign}(\mathbf{e}_j^*(x))}\,\mathbf{e}_j^*$. Then

$$a_m^*(x)\varphi_u(m) \le \mathbf{B}_m \frac{m\,a_m^*(x)}{\varphi_u^*(m)} \le \mathbf{B}_m \frac{\sum_{j\in G}|\mathbf{e}_j^*(x)|}{\|f^*\|} = \mathbf{B}_m \frac{f^*(x)}{\|f^*\|} \le \mathbf{B}_m\|x\|. \qquad \square$$

Theorem 4.4.2. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. For all $m \in \mathbb{N}$ we have

$$\mathbf{L}_m \le k_{2m}^c + \mathbf{B}_m.$$

Proof. Let $x \in \mathbb{X}$, $m \in \mathbb{N}$ and $G \subseteq \operatorname{supp}(x)$ of cardinality m such that $G_m(x) = P_G(x)$. Let $A \subseteq \mathbb{N}$ with $|A| \le m$ and $z = \sum_{j\in A} a_j \mathbf{e}_j$. We have

$$\|x - G_m(x)\| = \|(x-z) - P_{A\cup G}(x-z) + P_{A\setminus G}(x)\| \le \|(x-z) - P_{A\cup G}(x-z)\| + \|P_{A\setminus G}(x)\|.$$

Since $|A \cup B| \le 2m$,

$$\|(x-z) - P_{A\cup G}(x-z)\| \le k_{2m}^c\,\|x-z\|.$$

Let $|A \setminus G| \le |G \setminus A| =: r$. Invoking Corollary 1.3.3 and Lemma 4.4.1,

$$\begin{aligned}
\|P_{A\setminus G}(x)\| &\le \max_{j\in A\setminus G} |\mathbf{e}_j^*(x)|\varphi_u(r) \le \min_{j\in G\setminus A} |\mathbf{e}_j^*(x)|\varphi_u(r) \\
&= \min_{j\in G\setminus A} |\mathbf{e}_j^*(x-z)|\varphi_u(r) \le a_r^*(x-z)\varphi_u(r) \\
&\le \mathbf{B}_r\,\|x-z\|.
\end{aligned}$$

Taking into account that $r \le m$ and that $(\mathbf{B}_m)_{m=1}^\infty$ is non-decreasing we get

$$\|P_{A\setminus G}(x)\| \le \mathbf{B}_m\,\|x-z\|.$$

Combining we obtain the desired result. $\square$

Remark 4.4.3. Theorem 4.4.2 is optimal in all of the examples presented in Section 4.3. The inconvenience of this result is that we need to compute the unconditional parameter k_m and in Theorem 4.2.16 it is not necessary.

4.5 Open questions

Related to the new parameter $\overline{T}_m(\varphi_u, \varphi_u^*)$, we propose the following problems.

Question 1: characterize the systems $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ for which $\overline{T}_m(\varphi_u, \varphi_u^*) \lesssim \ln(m+1)$. All the examples we have tested seem to satisfy this property.

Question 2: characterize the systems for which $\max\{\mathbf{L}_m, \mathbf{L}_m^*\} \lesssim \ln(m+1)$. It is a sufficient condition that $\overline{T}_m(\varphi_u, \varphi_u^*) \lesssim \ln(m+1)$, but we do not know whether it is necessary.

Chapter 5

Thresholding Chebyshev Greedy Algorithm and Semi-greedy bases

As we have studied in Chapter 4, there are examples of almost-greedy bases that are conditional. In these cases, the parameter $\mathbf{L}_m$ may be as large as $O(\log(m))$, and the logarithm can be attained (Subsection 4.3.2). Here, we study the enhancement of the TGA which improves the rate of convergence as S. J. Dilworth, N. J. Kalton and D. Kutzarova introduced in [31]. These authors proved in [31] the following result:

Theorem 5.0.1 ([31]). Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$. $\mathscr{B}$ is almost-greedy if and only if for every $\lambda > 1$, there exists a constant C such that

$$\|x - \mathscr{G}_{\lceil \lambda m \rceil}(x)\| \leq \frac{C}{\lambda - 1}\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

When $\lambda \to 1^+$, this result does not produce a good bound. However, the authors of [31] propose the following alternative strategy:

Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$ and $A_m(x)$ be the m-th greedy set of $x \in \mathbb{X}$. Define the *m-th Chebyshev-greedy sum* as any element $\mathfrak{CG}_m(x) \in \langle \mathbf{e}_i : i \in A_m(x)\rangle$ such that

$$\|x - \mathfrak{CG}_m(x)\| = \min\left\{\left\| x - \sum_{n \in A_m(x)} a_n \mathbf{e}_n \right\| : a_n \in \mathbb{F}\right\}.$$

The collection $\{\mathfrak{CG}_m\}_{m=1}^\infty$ is called the *Thresholding Chebyshev Greedy Algorithm* (TCGA).

Definition 5.0.2 ([31]). An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *semi-greedy* if there exists a positive constant C such that

$$\|x - \mathfrak{CG}_m(x)\| \leq C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{5.1}$$

The least constant C in (5.1) is denoted by $C_{sg}[\mathscr{B}, \mathbb{X}] := C_{sg}$ and we will say that $\mathscr{B}$ is C_{sg}*-semi-greedy*.

S. J. Dilworth et al. proved the following result.

Theorem 5.0.3 ([31, Theorem 3.2]). Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

- If $\mathscr{B}$ is almost-greedy, then $\mathscr{B}$ is semi-greedy.

- If $\mathbb{X}$ has finite cotype and $\mathscr{B}$ is a Schauder semi-greedy basis, then $\mathscr{B}$ is almost-greedy.

This last result shows that it is possible to improve the rate of convergence using almost-greediness but, we need the condition of finite cotype over the space $\mathbb{X}$ and Schauder bases. In this chapter we will present the same characterization but we remove the condition of finite cotype and we will work with a more general condition than Schauder. This result is presented in [15, 22]. Also, we define the Lebesgue-type parameter for the Thresholding Chebyshev Greedy algorithm $\mathbf{L}_m^{\mathrm{ch}}$ and we study some bounds for this parameter as we have done for $\mathbf{L}_m$ in Chapter 4.

Throughout this chapter, we will use in some occasions the concept of *Cesàro basis*, that is, $\sup_m \|F_m\| < \infty$, where $F_m := \frac{1}{m}\sum_{n=1}^m P_n$ is the *m-th Cesàro operator* and $P_n(\cdot)$ is the n-th partial sum. In this case we use the constant

$$\beta = \max\big\{\sup_m \|F_m\|, \sup_m \|\mathrm{Id}_{\mathbb{X}} - F_m\|\big\}, \tag{5.2}$$

and we will say that $\mathscr{B}$ is a β-Cesàro basis. Of course, if a basis is Schauder then it is Cesàro and the converse is false since the trigonometric system in L_1 is Cesàro but not Schauder. Also, we remind the following notation: for a finite set A, $\max A = \max_{i\in A} i$, $\min A = \min_{j\in A} j$, and $A < B$ means that $\max A < \min B$. The results of this chapter are in [15, 20, 22].

5.1 ρ-admissibility

In this section, we define a new condition that can be found in [20]. This condition on the system $\mathscr{B}$ is slightly weaker than the Schauder or Cesàro conditions, but we introduce it because the theorems hold in these generality.

Definition 5.1.1. Given $\rho \geq 1$, an SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is *ρ-admissible* if the following holds: for each finite set $A \subset \mathbb{N}$, there exists $n_0 = n_0(A)$ such that, for all sets B with $\min B \geq n_0$ and $|B| = |A|$,

$$\Big\|\sum_{n\in A} \alpha_n \mathbf{e}_n\Big\| \leq \rho \Big\|\sum_{n\in A\cup B} \alpha_n \mathbf{e}_n\Big\|, \quad \forall\, \alpha_n \in \mathbb{F} \tag{5.3}$$

We define the set $\mathscr{R}(A) = \{B : (5.3) \text{ is satisfied}\}$. Observe that, if $B \in \mathscr{R}(A)$ with $A \cap B = \emptyset$, (5.3) implies that

$$\|\sum_{n\in B} \alpha_n \mathbf{e}_n\| \leq (\rho+1)\|\sum_{n\in A\cup B} \alpha_n \mathbf{e}_n\|, \forall \alpha_n \in \mathbb{F}.$$

We now give some general conditions in $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ and $\mathbb{X}$ under which ρ-admissibility holds. We recall a few standard definitions (see e.g. [48]). A sequence $(\mathbf{e}_n)_{n=1}^\infty$ is *weakly null* if

$$\lim_{n\to\infty} x^*(\mathbf{e}_n) = 0, \quad \forall\, x^* \in \mathbb{X}^*.$$

Given a subset $Y \subset \mathbb{X}^*$, we shall say that $(\mathbf{e}_n)_{n=1}^\infty$ is *Y-null* if

$$\lim_{n\to\infty} y(\mathbf{e}_n) = 0, \quad \forall\, y \in Y.$$

Given $\lambda \in (0,1]$, we say that a set $Y \subset \mathbb{X}^*$ is λ*-norming* whenever

$$\sup_{x^* \in Y, \|x^*\| \leq 1} |x^*(x)| \geq \lambda \, \|x\|, \quad \forall \, x \in \mathbb{X}.$$

Proposition 5.1.2. Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be a biorthogonal system in $\mathbb{X} \times \mathbb{X}^*$. Suppose that the sequence $\{\tilde{\mathbf{e}}_n := \|\mathbf{e}_n^*\| \, \mathbf{e}_n\}_{n=1}^\infty \subset \mathbb{X}$ is Y-null, for some subset $Y \subset \mathbb{X}^*$ which is λ-norming. Then $(\mathbf{e}_n)_{n=1}^\infty$ is ρ-admissible for every $\rho > 1/\lambda$.

Proof. Consider a finite set $A \subset \mathbb{N}$ with say $|A| = m$ and denote $E := [\mathbf{e}_n]_{n \in A}$. Given $\varepsilon > 0$, one can find a finite set $S \subset Y \cap \{x^* \in \mathbb{X}^* : \|x^*\| = 1\}$ so that

$$\max_{x^* \in S} |x^*(e)| \geq (1-\varepsilon)\lambda \|e\|, \quad \forall \, e \in E. \tag{5.4}$$

Indeed, it suffices to verify the above inequality for e of norm 1. Pick an $\varepsilon\lambda/2$-net $(z_k)_{k=1}^N$ in the unit sphere of E. For any k find a norm one $z_k^* \in Y$ so that $|z_k^*(z_k)| > (1-\varepsilon/2)\lambda$. We claim that $S = \{z_k^* : 1 \leq k \leq N\}$ has the desired properties. To see this, pick a norm one $e \in E$, and find k with $\|e - z_k\| \leq \varepsilon\lambda/2$. Then

$$\max_{x^* \in S} |x^*(e)| \geq |z_k^*(e)| \geq |z_k^*(z_k)| - \|e - z_k\| \geq (1-\varepsilon/2)\lambda - \varepsilon\lambda/2 = (1-\varepsilon)\lambda.$$

Next, since the sequence $\{\|\mathbf{e}_n^*\| \, \mathbf{e}_n\}$ is Y-null, for each $\delta > 0$ we can find an integer $n_0 > \max A$ so that

$$\max_{x^* \in S} |x^*(\mathbf{e}_n)| \, \|\mathbf{e}_n^*\| \leq \frac{\delta\lambda}{m}, \quad \forall \, n \geq n_0.$$

Pick any B of cardinality m with $\min B \geq n_0$, and let $G := [\mathbf{e}_n]_{n \in B}$. For $f = \sum_{n \in B} \mathbf{e}_n^*(f) \mathbf{e}_n \in G$, we have

$$\max_{x^* \in S} |x^*(f)| \leq \max_{x^* \in S} \sum_{n \in B} |x^*(\mathbf{e}_n)| \, \|\mathbf{e}_n^*\| \, \|f\| \leq \delta\lambda \|f\|. \tag{5.5}$$

We claim that

$$\|e + f\| \geq \frac{(1-\varepsilon-\delta)\lambda}{1+\delta\lambda} \|e\|, \text{ for any } e \in E, f \in G. \tag{5.6}$$

To show this, we fix $\gamma > 0$ (to be chosen later), and assume first that $\|f\| \geq (1+\gamma)\|e\|$. Then,

$$\|e + f\| \geq \|f\| - \|e\| \geq \gamma \|e\|.$$

Next assume that $\|f\| < (1+\gamma)\|e\|$; then using (5.4) and (5.5) we obtain that

$$\|e + f\| \geq \max_{x^* \in S} |x^*(e+f)| \geq (1-\varepsilon)\lambda \|e\| - \delta\lambda \|f\| > (1 - \varepsilon - \delta(1+\gamma))\lambda \|e\|.$$

We now choose γ so that $\gamma = (1 - \varepsilon - \delta(1+\gamma))\lambda$, that is,

$$\gamma = \frac{(1-\varepsilon-\delta)\lambda}{1+\delta\lambda},$$

which shows the claim in (5.6). Now, given $\rho > 1/\lambda$, we may pick $\delta = \varepsilon$ sufficiently small so that the above number $\gamma > 1/\rho$. Then, (5.6) becomes

$$\|e+f\| \geq \frac{1}{\rho}\|e\|, \quad \text{for any } e \in [e_n]_{n\in A},\, f \in [e_n]_{n\in B},$$

for all B with $\min B \geq n_0$ and $|B| = |A| = m$. Thus, $(\mathbf{e}_n)_{n=1}^{\infty}$ is ρ-admissible. $\square$

Remark 5.1.3. A few cases where the hypotheses in the above proposition can be applied are the following:

(1) When the sequence $(\tilde{\mathbf{e}}_n)_{n=1}^{\infty}$ is weakly null, since $Y = \mathbb{X}^*$ is always 1-norming.

(2) When $\sup_{n\geq 1}\|\mathbf{e}_n\|\,\|\mathbf{e}_n^*\|_* < \infty$ and $Y = [\mathbf{e}_n^*]_{n\in\mathbb{N}}$ is λ-norming, since the first condition implies that $(\tilde{\mathbf{e}}_n)_{n=1}^{\infty}$ is Y-null. In particular, when $(\mathbf{e}_n)_{n=1}^{\infty}$ is a Schauder basis in $\mathbb{X}$, in which case the above conditions hold with $\lambda = 1/\mathfrak{K}_b$ (see [72, Theorems I.3.1 and I.12.2]).

(3) In every separable Banach space $\mathbb{X}$, if one picks $(\mathbf{e}_n)_{n=1}^{\infty}$ to be an SM-basis with the properties in (2) and $\lambda = 1$ (see [73, Theorem III.8.5] for the existence of such bases).

(4) Let $\mathbb{X} = C(K)$ where K is a compact Hausdorff set and let μ be a Radon probability measure in K with $\operatorname{supp}\mu = K$. Then, the natural embedding of $C(K)$ into $L_\infty(\mu)$ is isometric, and therefore $Y = L_1(\mu)$ is 1-norming in $\mathbb{X}$. Let $(\mathbf{e}_n)_{n=1}^{\infty}$ be a complete system in $\mathbb{X}$ which is orthonormal with respect to μ and uniformly bounded, that is, $\int_K \mathbf{e}_n\overline{\mathbf{e}_m}\,d\mu = \delta_{n,m}$ and $\sup_n\|\mathbf{e}_n\|_\infty < \infty$. Then the sequence $(\mathbf{e}_n)_{n=1}^{\infty}$ is $L_1(\mu)$-null in $\mathbb{X}$. Indeed, this follows from case (2), and the fact that $C(K)$ is dense in $L_1(\mu)$.

   Examples of such systems in $C(K)$ include the trigonometric system in $C[0,1]$ (in the real or complex case), as well as certain polygonal versions of the Walsh system [28, 69, 83], or any reorderings of them (which may cease to be Cesàro bases).

(5) As a dual of the previous, if $\mathbb{X} = L_1(\mu)$ then every system $(\mathbf{e}_n)_{n=1}^{\infty}$ as in (4) is weakly null, and hence case (1) applies.

5.2 Semi-greedy bases

The main theorem of this section is the following:

Theorem 5.2.1. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_q-quasi-greedy and C_s-super-democratic, then $\mathscr{B}$ is C_{sg}-semi-greedy with constant
$$C_{sg} \leq C_q + 4C_qC_s.$$

b) If $\mathscr{B}$ is C_{sg}-semi-greedy and ρ-admissible, then $\mathscr{B}$ is C_s-super-democratic and C_q-quasi-greedy with constants
$$C_s \leq C_{sg}^2\rho(1+\rho), \quad C_q \leq C_{sg}\rho(1+(1+\rho)C_{sg}).$$

The proof of this theorem for Schauder bases can be found in [15], and the proof using the ρ-admissibility can be found in [22]. The techniques used in both cases are similar.

Remark 5.2.2. S. J. Dilworth et al. ([36]) proved a) with the bound $C_{sg} = O(C_q^3 C_d)$, where C_d is the democracy constant. Here, we show that $C_{sg} = O(C_q C_s)$, which improves the order of the constants since $C_s \leq 4\kappa^2 C_d C_q$, where κ is 1 if $\mathbb{F} = \mathbb{R}$ or 2 if $\mathbb{F} = \mathbb{C}$. To exhibit this fact, we only have to use c) of Proposition 2.2.5.

Corollary 5.2.3. If $\mathscr{B}$ is an SM-Schauder basis in $\mathbb{X}$, $\mathscr{B}$ is almost-greedy if and only if $\mathscr{B}$ is semi-greedy.

Proof. The proof follows using Theorem 5.2.1, Theorem 2.5.4 and Remark 5.1.3. □

Proof of Theorem 5.2.1. First, we show the proof of a). Suppose that $\mathscr{B}$ is C_q-quasi-greedy and C_s-super-democratic. To show the semi-greediness, we will follow the same procedure as in the proof of [35, Theorem 4.1] and [31, Theorem 3.2]. Take $x \in \mathbb{X}$ and for each $\delta > 0$, take $z = \sum_{i\in B} a_i \mathbf{e}_i$ with $|B| \leq m$ such that $\|x - z\| < \sigma_m(x) + \delta$. Let $\mathscr{G}_m(x)$ the m-th greedy sum of x with $A = \operatorname{supp}(\mathscr{G}_m(x))$. We write $x - z := \sum_{i=1}^{\infty} y_i e_i$, where $y_i = \mathbf{e}_i^*(x) - a_i$ for $i \in B$ and $y_i = \mathbf{e}_i^*(x)$ for $i \notin B$. To prove that $\mathscr{B}$ is semi-greedy we only have to show that there exists $w \in \mathbb{X}$ so that $\operatorname{supp}(x - w) \subset A$ and $\|w\| \leq c\|x - z\|$ for some positive constant c. If $\alpha = \max_{j\notin A} |\mathbf{e}_j^*(x)|$, we take the element w as is defined in [31]:

$$w := \sum_{i\in A} T_\alpha(y_i)\mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x) = \sum_{i=1}^{\infty} T_\alpha(y_i)\mathbf{e}_i + \sum_{i\in B\setminus A} (\mathbf{e}_i^*(x) - T_\alpha(y_i))\mathbf{e}_i.$$

Of course, w satisfies that $\operatorname{supp}(x - w) \subset A$ and we will prove that

$$\|w\| \leq (C_q + 4C_q C_s)\|x - z\|.$$

To obtain this bound, using Lemma 2.1.13,

$$\left\| \sum_{i=1}^{\infty} T_\alpha(y_i)\mathbf{e}_i \right\| \leq C_q \|x - z\|. \tag{5.7}$$

Taking into account that $|\mathbf{e}_i^*(x) - T_\alpha(y_i)| \leq 2\alpha$ for $i \in B \setminus A$, using (i) of Corollary 1.3.3,

$$\left\| \sum_{i\in B\setminus A} (\mathbf{e}_i^*(x) - T_\alpha(y_i))\mathbf{e}_i \right\| \leq 2\alpha \sup_{\eta\in\Psi_{B\setminus A}} \|\mathbf{1}_{\eta(B\setminus A)}\| \leq 2 \min_{j\in A\setminus B} |\mathbf{e}_j^*(x - z)| \sup_{\eta\in\Psi_{B\setminus A}} \|\mathbf{1}_{\eta(B\setminus A)}\|. \tag{5.8}$$

Now, based on ([44, Lemma 2.1]), we can find a greedy set Γ of $x - z$ with the following conditions:

- $|\Gamma| = |B \setminus A|$,
- $\min_{j\in A\setminus B} |\mathbf{e}_j^*(x - z)| \leq \min_{j\in\Gamma} |\mathbf{e}_j^*(x - z)|$.

Hence, if $\varepsilon = \{\operatorname{sign}(\mathbf{e}_j^*(x - z))\}$, applying the super-democracy,

$$\sup_{\eta\in\Psi_{B\setminus A}} \|\mathbf{1}_{\eta(B\setminus A)}\| \leq C_s \|\mathbf{1}_{\varepsilon\Gamma}\|. \tag{5.9}$$

Using (5.9) in (5.8),

$$\left\| \sum_{i \in B \setminus A} (\mathbf{e}_i^*(x) - T_\alpha(y_i))\mathbf{e}_i \right\| \leq 2C_s \min_{j \in \Gamma} |\mathbf{e}_j^*(x-z)| \|\mathbf{1}_{\varepsilon\Gamma}\|. \tag{5.10}$$

Now, by Lemma 2.1.14,

$$\min_{j \in \Gamma} |\mathbf{e}_j^*(x-z)| \|\mathbf{1}_{\varepsilon\Gamma}\| \leq 2C_q \|x-z\|. \tag{5.11}$$

Thus, using (5.7), (5.10), (5.11), the basis is C_{sg}-semi-greedy with constant $C_{sg} \leq (C_q + 4C_qC_s)$.

Now, we prove b). First, we show the super-democracy. Take two finite sets $|A| \leq |B|$, $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$ and show (2.16). Since the basis is ρ-admissible, we can find a set $F \in \mathscr{R}(A \cup B)$ such that $|F| = |A \cup B|$ and $F > A \cup B$. Hence,

$$\left\| \sum_{n \in A \cup B} \alpha_n \mathbf{e}_n \right\| \leq \rho \left\| \sum_{n \in A \cup B \cup F} \alpha_n \mathbf{e}_n \right\|, \forall \alpha_n \in \mathbb{F}. \tag{5.12}$$

Select a set $C \subset F$ such that $|C| = |A|$. Consider the element $x := \mathbf{1}_{\varepsilon A} + (1+\delta)\mathbf{1}_C$, with $\delta > 0$. Using the TCGA, there exists a collection $(a_i)_{i \in C}$ such that

$$x - \mathfrak{C}\mathfrak{G}_m(x) = \mathbf{1}_{\varepsilon A} + \sum_{i \in C} a_i \mathbf{e}_i.$$

Using semi-greediness and (5.12) with $\alpha_n = \varepsilon_n$ if $n \in A$, $\alpha_n = 0$ if $n \in (B \setminus A) \cup (F \setminus C)$ and $\alpha_n = a_n$ if $n \in C$,

$$\|\mathbf{1}_{\varepsilon A}\| \leq \rho \left\| \mathbf{1}_{\varepsilon A} + \sum_{i \in C} a_i \mathbf{e}_i \right\| \leq C_{sg}\rho \|x - \mathbf{1}_{\varepsilon A}\| = C_{sg}\rho \|(1+\delta)\mathbf{1}_C\|.$$

Taking the limit when δ goes to 0, we obtain that

$$\|\mathbf{1}_{\varepsilon A}\| \leq C_{sg}\rho \|\mathbf{1}_C\|. \tag{5.13}$$

Now, consider $y := (1+\delta)\mathbf{1}_{\eta B} + \mathbf{1}_C$ with $\delta > 0$. Applying the TGCA, there exists a sequence $(b_i)_{i \in B}$ such that

$$y - \mathfrak{C}\mathfrak{G}_m(y) = \sum_{i \in B} b_i \mathbf{e}_i + \mathbf{1}_C.$$

As before, using semi-greediness and (5.12) with $\alpha_n = 0$ if $n \in (A \setminus B) \cup (F \setminus C)$, $\alpha_n = b_n$ if $n \in B$ and $\alpha_n = 1$ if $n \in C$,

$$\|\mathbf{1}_C\| \leq (1+\rho) \left\| \mathbf{1}_C + \sum_{i \in B} b_i \mathbf{e}_i \right\| \leq C_{sg}(1+\rho)\|x - \mathbf{1}_C\| = C_{sg}(1+\rho)\|(1+\delta)\mathbf{1}_{\eta B}\|.$$

Taking the limit when δ goes to 0 and using (5.13), we obtain that

$$\|\mathbf{1}_{\varepsilon A}\| \leq C_{sg}^2 \rho(1+\rho)\|\mathbf{1}_{\eta B}\|.$$

Thus, the basis is C_s-super-democratic with $C_s \le C_{sg}^2\rho(1+\rho)$.

Now, we prove quasi-greediness. Take $x \in \mathbb{X}$ with finite support $B = \mathrm{supp}(x)$. Using the ρ-admissibility, we can find a set $C \in \mathscr{R}(B)$ such that $|C| = |B|$ and $C > B$.

Take $m \in \mathbb{N}$ and $\delta > 0$. Define the element $y := (x - \mathscr{G}_m(x)) + (\alpha+\delta)\mathbf{1}_F$, where $F \subset C$ with $|F| = m$ and $\alpha = \min_{j\in A} |\mathbf{e}_j^*(x)|$, where A is the m-th greedy set of x. Then, using the TCGA, we find a sequence $(a_i)_{i\in F}$ such that

$$y - \mathfrak{CG}_m(y) = (x - \mathscr{G}_m(x)) + \sum_{i\in F} a_i \mathbf{e}_i.$$

Using semi-greediness and the ρ-admissibility with $\alpha_n = 0$ if $n \in A \cup (C \setminus F)$, $\alpha_n = \mathbf{e}_n^*(x)$ if $n \in A^c \cap B$ and $\alpha_n = a_n$ if $n \in F$,

$$\|x - \mathscr{G}_m(x)\| \le \rho\|y - \mathfrak{CG}_m(y)\| \le \rho C_{sg}\sigma_m(y) \le C_{sg}\rho(\|x\| + \|(\alpha+\delta)\mathbf{1}_F\|).$$

Taking the limit when δ goes to 0, $\|x - \mathscr{G}_m(x)\| \le C_{sg}\rho(\|x\| + \|\alpha\mathbf{1}_F\|)$. Now, take $\eta \equiv \{\mathrm{sign}(\mathbf{e}_i^*(x))\}$ and define $z := \sum_{i\in A}(\mathbf{e}_i^*(x) + \delta\eta_i)\mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x) + \alpha\mathbf{1}_F$ with $\delta > 0$. Thus, by TCGA, there exists a collection $(b_i)_{i\in A}$ such that

$$z - \mathfrak{CG}_m(z) = \sum_{i\in A} b_i\mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x) + \alpha\mathbf{1}_F.$$

Again, using semi-greediness and the ρ-admissibility with $\alpha_n = b_n$ if $n \in A$, $\alpha_n = \mathbf{e}_n^*(x)$ if $n \in A^c \cap B$, $\alpha_n = \alpha$ if $n \in F$ and $\alpha_n = 0$ if $n \in C \setminus F$,

$$\begin{aligned} \|\alpha\mathbf{1}_F\| &\le (1+\rho)\left\|\sum_{i\in A} b_i\mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x) + \alpha\mathbf{1}_F\right\| \le C_{sg}\sigma_{|F|}(z) \\ &\le C_{sg}(1+\rho)\|z - \alpha\mathbf{1}_F\| \\ &= C_{sg}(1+\rho)\left\|\sum_{i\in A}(\mathbf{e}_i^*(x) + \delta\eta_i)\mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x)\right\|. \end{aligned}$$

Taking limits when δ goes to 0, we obtain that $\mathscr{B}$ is C_q-quasi-greedy with $C_q \le C_{sg}\rho(1+(1+\rho)C_{sg})$ for elements with finite support. To prove the same bound for general elements, we proceed as follows: define now $C_1 = C_{sg}\rho(1+(1+\rho)C_{sg})$ and take $x \in \mathbb{X}$. By Lemma 2.0.3, if $A = \mathrm{supp}(\mathscr{G}_m(x))$, taking $\varepsilon' \le \frac{\varepsilon}{1+C_1+\|P_A\|}$ with $\varepsilon > 0$, there exists $y \in \mathbb{X}$ with finite support such that $\|x-y\| < \varepsilon'$ and $A = \mathrm{supp}(\mathscr{G}_m(y))$. Hence,

$$\begin{aligned} \|x - \mathscr{G}_m(x)\| &\le \|x-y\| + \|y - \mathscr{G}_m(y)\| + \|\mathscr{G}_m(y) - \mathscr{G}_m(x)\| \\ &\le \varepsilon' + C_1\|y\| + \|P_A(x-y)\| \\ &\le \varepsilon' + C_1\varepsilon' + C_1\|x\| + \|P_A\|\varepsilon' \\ &= \varepsilon'(1 + C_1 + \|P_A\|) + C_1\|x\| \\ &\le \varepsilon + C_1\|x\|. \end{aligned}$$

Thus, $\mathscr{B}$ is C_q-quasi-greedy with $C_q \le C_1$. □

5.3 Lebesgue-type parameter for the TCGA

As for the TGA, if an SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is not semi-greedy, we want to study how far is the basis from been semi-greedy. For that, we recall some notation: if $A \in \mathscr{G}(x,m)$, a *Chebyshev greedy operator of order m* is any $\mathfrak{CG}_m(x) \in \langle \mathbf{e}_n : n \in A\rangle$ such that

$$\|x - \mathfrak{CG}_m(x)\| = \min\left\{ \left\| x - \sum_{i\in A} a_i \mathbf{e}_i \right\| : (a_i)_i \in \mathbb{F} \right\}.$$

We write $\mathbb{G}_m^{\mathrm{ch}}$ for the set of all Chebyshev greedy operators or order m and $\mathbb{G}^{\mathrm{ch}} = \cup_{m\geq 1}\mathbb{G}_m^{\mathrm{ch}}$.

Then, to quantify the performance of Chebyshev greedy operators as m-term approximations, for every $m = 1,2,...$, we define the smallest parameter $\mathbf{L}_m^{\mathrm{ch}}[\mathscr{B},\mathbb{X}] := \mathbf{L}_m^{\mathrm{ch}}$ such that

$$\|x - \mathfrak{CG}_m(x)\| \leq \mathbf{L}_m^{\mathrm{ch}} \sigma_m(x), \quad \forall x \in \mathbb{X}, \quad \forall \mathfrak{CG}_m \in \mathbb{G}_m^{\mathrm{ch}}.$$

The first time that the parameter $\mathbf{L}_m^{\mathrm{ch}}$ was studied was in the paper [35] for quasi-greedy bases and here we focus our attention to study the boundedness of $\mathbf{L}_m^{\mathrm{ch}}$ for general bases. Also, one may compare the bounds for $\mathbf{L}_m^{\mathrm{ch}}$ with those for $\mathbf{L}_m$ given in Chapter 4:

$$(1)\ \mathbf{L}_m \leq 1 + 3\mathfrak{K}m, \qquad (2)\ \mathbf{L}_m \leq k_{2m}^c + \tilde{g}_m \tilde{\mu}_m, \qquad \text{and} \qquad (3)\ \mathbf{L}_m \geq \tilde{\mu}_m^d.$$

Observe that one also has the trivial inequalities

$$\mathbf{L}_m^{\mathrm{ch}} \leq \mathbf{L}_m \leq k_m^c \mathbf{L}_m^{\mathrm{ch}}.$$

Indeed, $\mathbf{L}_m^{\mathrm{ch}} \leq \mathbf{L}_m$ is direct by definition, while $\mathbf{L}_m \leq k_m^c \mathbf{L}_m^{\mathrm{ch}}$ can be proved as follows: take $x \in \mathbb{X}$ and $A = \mathrm{supp}(\mathscr{G}_m(x))$. Pick a Chebyshev greedy operator $\mathfrak{CG}_m$ such that $\mathrm{supp}(\mathfrak{CG}_m(x)) = A$. Then

$$\|x - \mathscr{G}_m(x)\| = \|(\mathrm{Id}_{\mathbb{X}} - P_A)(x)\| = \|(\mathrm{Id}_{\mathbb{X}} - P_A)(x - \mathfrak{CG}_m(x))\| \leq k_m^c \|x - \mathfrak{CG}_m(x)\|,$$

so $\mathbf{L}_m \leq k_m^c \mathbf{L}_m^{\mathrm{ch}}$. Hence, when $\mathscr{B}$ is unconditional, then $\mathbf{L}_m \approx \mathbf{L}_m^{\mathrm{ch}}$. However for all conditional quasi-greedy and democratic bases we have $\mathbf{L}_m^{\mathrm{ch}} = O(1)$ (see Theorem 5.2.1), but $\mathbf{L}_m \to \infty$.

To study upper and lower bounds of $\mathbf{L}_m^{\mathrm{ch}}$ for general bases, we will use the parameters introduced in Chapter 4 and also the following new parameters:

- Given an integer $c \geq 2$, we define

$$\vartheta_{m,c} := \sup\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} : \varepsilon \in \Psi_A, \eta \in \Psi_B,\ |A| = |B| \leq m \ \text{ with } A > cB \text{ or } B > cA \right\}. \quad (5.14)$$

  Alternatively, we can define $\vartheta_{m,c} = \sup_{|A|\leq m} \vartheta_c(A)$, where

$$\vartheta_c(A) = \sup_{\substack{B:|B|=|A| \\ B>cA \\ \varepsilon\in\Psi_A,\eta\in\Psi_B}} \max\left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \right\},$$

and finally, define

$$\vartheta_m = \sup_{|A|\le m} \inf_{c\ge 1} \vartheta_c(A). \tag{5.15}$$

5.3.1 Upper bounds of $\mathbf{L}_m^{\mathrm{ch}}$

The following results are all proved in [20].

Theorem 5.3.1. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$. Then,

$$\mathbf{L}_m^{\mathrm{ch}} \le 1 + 2\mathfrak{K}m,$$

where $\mathfrak{K} = \sup_{n,j} \|\mathbf{e}_n\| \|\mathbf{e}_j^*\|_*$.

Proof. Let $x \in \mathbb{X}$ and $\mathfrak{CG}_m \in \mathbb{G}_m^{\mathrm{ch}}$ be a fixed Chebyshev greedy operator, and denote by $A = \operatorname{supp}(\mathfrak{CG}_m(x)) \in \mathscr{G}(x,m)$. Pick any $z = \sum_{n\in B} b_n\mathbf{e}_n$ such that $|B| \le m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m(x)\| \le \|x - P_{A\cap B}(x)\| \le \|P_{B\setminus A}(x)\| + \|x - P_B(x)\|.$$

$$\|P_{B\setminus A}(x)\| \le \sup_n \|\mathbf{e}_n\| \sum_{j\in B\setminus A} |\mathbf{e}_j^*(x)| \le \sup_n \|\mathbf{e}_n\| \sum_{j\in A\setminus B} |\mathbf{e}_j^*(x-z)| \le \mathfrak{K}m\|x-z\|.$$

On the other hand, using the inequality (4.18),

$$\|x - P_B(x)\| = \|(\mathrm{Id}_{\mathbb{X}} - P_B)(x-z)\| \le k_m^c \|x-z\| \le (1+\mathfrak{K}m)\|x-z\|.$$

Hence, $\mathbf{L}_m^{\mathrm{ch}} \le 1 + 2\mathfrak{K}m$.

□

Example 5.3.2. To study the optimality of this result, we will use the Examples 4.1.2 and 4.3.1. We remind the definitions: the difference basis is the collection $(\mathbf{x}_n)_{n=1}^\infty$,

$$\mathbf{x}_1 = \mathbf{e}_1\,, \quad \mathbf{x}_n = \mathbf{e}_n - \mathbf{e}_{n-1}\,,\; n = 2,3,\ldots,$$

and $(\mathbf{e}_n)_{n=1}^\infty$ denotes the canonical basis in $\ell_1(\mathbb{N})$. This is a monotone basis in $\mathbb{X} = \ell_1$ and for finitely supported real scalars $(b_n)_{n=1}^\infty$ one has

$$\Big\| \sum_{n=1}^\infty b_n \mathbf{x}_n \Big\| = \sum_{n=1}^\infty |b_n - b_{n+1}|\,.$$

In particular, $\|\mathbf{x}_1\| = 1$ and $\|\mathbf{x}_n\| = 2$ if $n \ge 2$. The dual system consists of the ℓ_∞-vectors $\mathbf{x}_n^* = \sum_{m=n}^\infty \mathbf{e}_m^*$, so for $\{c_n\} \in c_{00}$ it holds that

$$\Big\| \sum_{n=1}^\infty c_n \mathbf{x}_n^* \Big\|_* = \sup_{n\ge 1} \Big| \sum_{j=1}^n c_j \Big|\,.$$

The system $(\mathbf{x}_n^*)_{n=1}^\infty$ is the summing basis and $\|\mathbf{x}_n^*\|_* = 1$ for all n. Then, $\mathfrak{K} = 2$ in Theorem 5.3.1.

To show that $\mathbf{L}_m^{\mathrm{ch}} = 1+4m$ for the difference basis, consider the vector $x = \sum_n b_n y_n$ with coefficients (b_n) given by

$$\Big(1, \underbrace{1,1,-1,1}, ..., \underbrace{1,1,-1,1}, 0, ...\Big),$$

where the block $(1,1,-1,1)$ is repeated m times. If we take $\Gamma = \{2,6,...,4m-2\}$ as a greedy set for x of cardinality m, then

$$\begin{aligned}
\|x - \mathfrak{CG}_m(x)\| &= \inf_{(a_j)_{j=1}^m} \|x - \sum_{j=1}^m a_j y_{4j-2}\| \\
&= \inf_{(a_j)_{j=1}^m} \left\| \Big(1, 1-a_1, 1, -1, 1, ..., 1-a_m, 1, -1, 1, 0, ...\Big) \right\| \\
&= \inf_{(a_j)_{j=1}^m} 2\sum_{j=1}^m |a_j| + 4m + 1 = 4m+1.
\end{aligned}$$

Hence, in this case we have $\mathfrak{CG}_m(x) = 0$. On the other hand,

$$\sigma_m(x) \le \big\|x + 2\sum_{j=1}^m y_{4j}\big\| = \|(1,1,1,1,1,...,1,1,1,1,0,...)\| = 1.$$

This shows that $\mathbf{L}_m^{\mathrm{ch}} \ge 1+4m$ and using Theorem 5.3.1 with $\mathfrak{K} = 2$ we have the equality.

Now, for the summing basis take the following element:

$$x = \Big(\underbrace{\frac{1}{2}, 1, \frac{1}{2}}, ..., \underbrace{\frac{1}{2}, 1, \frac{1}{2}}; \frac{1}{2}; \underbrace{-1, 1}, ..., \underbrace{-1, 1}, 0, ...\Big),$$

where we have m blocks of $\left(\frac{1}{2}, 1, \frac{1}{2}\right)$ and m blocks of $(-1,1)$. Picking $A = \{n : x_n = -1\}$ as a greedy set of x of size m, we see that

$$\begin{aligned}
\|x - \mathfrak{CG}_m(x)\| &= \min_{a_i, i=1,...,m} \left\| \Big(\frac{1}{2}, 1, \frac{1}{2}, ..., \frac{1}{2}, 1, \frac{1}{2}; \frac{1}{2}; a_1, 1, a_2, 1, ..., a_m, 1, 0, ...\Big) \right\| \\
&\ge \left\| \Big(\frac{1}{2}, 1, \frac{1}{2}, ..., \frac{1}{2}, 1, \frac{1}{2}; \frac{1}{2}; 0, ...\Big) \right\| = 2m + \frac{1}{2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma_m(x) &\le \left\| x - 2(0,1,0,...,0,1,0;0,...) \right\| \\
&= \left\| \Big(\frac{1}{2}, -1, \frac{1}{2}, ..., \frac{1}{2}, -1, \frac{1}{2}; \frac{1}{2}; -1, 1, ..., -1, 1, 0...\Big) \right\| = \frac{1}{2}.
\end{aligned}$$

Hence, $\mathbf{L}_m^{\mathrm{ch}} \ge 1+4m$ and we conclude that $\mathbf{L}_m^{\mathrm{ch}} = 1+4m$ by Theorem 5.3.1 since $\mathfrak{K} = 2$. Observe that we also have that $\mathfrak{CG}_m(x) = 0$.

Theorem 5.3.3. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. Then, for all $m \ge 1$,

$$\mathbf{L}_m^{\mathrm{ch}} \le g_{2m}^c + \min\big\{ \tilde{g}_m \tilde{\mu}_m,\, \gamma_{2m} \tilde{g}_{2m} \tilde{\mu}_m^d \big\}. \tag{5.16}$$

To show this result we need the following technical lemma.

Lemma 5.3.4. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be an SM-basis in a Banach space $\mathbb{X}$. Let $x \in \mathbb{X}$ and $\varepsilon = \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$. For every set finite $A \subset \mathbb{N}$, if $\alpha = \min_{n\in A} |\mathbf{e}_n^*(x)|$, then

$$\alpha \|\mathbf{1}_{\varepsilon A}\| \,\le\, \gamma_k \tilde{g}_k \|x\|, \tag{5.17}$$

where $k = |A \cup \Lambda_\alpha(x)|$ and $\Lambda_\alpha(x) = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$.

Proof. Call $G = A \cup \Lambda_\alpha(x)$, $k = |G|$, and notice that it is a greedy set for x. Then,

$$\alpha \|\mathbf{1}_{\varepsilon A}\| \,\le\, \alpha\, \gamma_k \|\mathbf{1}_{\varepsilon G}\| \,\le\, \gamma_k\, \tilde{g}_k\, \|x\|,$$

using Lemma 4.0.8 in the last step. □

Proof of Theorem 5.3.3: The scheme of the proof follows the lines in [31, Theorem 3.2], [35, Theorem 4.1] and our Theorem 5.2.1.

Given $x \in \mathbb{X}$ and $\mathfrak{CG}_m \in \mathbb{G}_m^{\mathrm{ch}}$, we denote by $A = \mathrm{supp}(\mathfrak{CG}_m(x))$. Pick any $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $|B| \le m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m(x)\| \le \|x - p\|, \quad \text{for any } p = \textstyle\sum_{n\in A} a_n \mathbf{e}_n. \tag{5.18}$$

We make the selection of p suggested in [31]. Namely, if $\alpha = \max_{n\notin A} |\mathbf{e}_n^*(x)|$, we let

$$p := P_A(x) - P_A\big(T_\alpha(x-z)\big).$$

It is easily verified that

$$\begin{aligned} x - p &= (\mathrm{Id}_{\mathbb{X}} - P_A)\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z) \\ &= P_{B\setminus A}\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z). \end{aligned} \tag{5.19}$$

Since $\Lambda_\alpha(x-z) = \{n : |\mathbf{e}_n^*(x-z)| > \alpha\} \subset A \cup B$, then Lemma 4.0.8 gives

$$\big\|T_\alpha(x-z)\big\| \le g_{2m}^c \|x-z\|. \tag{5.20}$$

Next we treat the first term in (5.19). Observe that $\max_{n\in B\setminus A} |\mathbf{e}_n^*(x - T_\alpha(x-z))| \le 2\alpha$, so Corollary 1.3.3 gives

$$\begin{aligned} \big\|P_{B\setminus A}\big(x - T_\alpha(x-z)\big)\big\| &\le 2\alpha \sup_{\varepsilon\in\Psi_{B\setminus A}} \|\mathbf{1}_{\varepsilon B\setminus A}\| \\ &\le 2 \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)| \sup_{\varepsilon\in\Psi_{B\setminus A}} \|\mathbf{1}_{\varepsilon B\setminus A}\| = (*). \end{aligned} \tag{5.21}$$

At this point we have two possible approaches. Let $\eta \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x-z))\}$. In the first approach we pick a greedy set $\Gamma \in \mathscr{G}(x-z, |A\setminus B|)$, and control (5.21) by

$$(*) \le 2 \min_{n\in\Gamma} |\mathbf{e}_n^*(x-z)|\, \tilde{\mu}_m \big\|\mathbf{1}_{\eta\Gamma}\big\| \,\le\, 2\,\tilde{\mu}_m\, \tilde{g}_m \|x-z\|, \tag{5.22}$$

using Lemma 4.0.8 in the last step. In the second approach, we argue as follows

$$(*) \leq 2 \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)|\, \tilde{\mu}_m^d \left\|\mathbf{1}_{\eta(A\setminus B)}\right\| \leq 2\gamma_{2m}\, \tilde{g}_{2m}\, \tilde{\mu}_m^d\, \|x-z\|, \tag{5.23}$$

using in the last step Lemma 5.3.4 and the fact that, if $\delta = \min_{A\setminus B} |\mathbf{e}_n^*(x-z)|$, then the set $(A\setminus B)\cup\{n : |\mathbf{e}_n^*(x-z)| > \delta\} \subset A\cup B$ and hence has cardinality less than or equal to $2m$.

We can now combine the estimates displayed in (5.18)-(5.23) and obtain

$$\|x - \mathfrak{CG}_m x\| \leq \left[g_{2m}^c + 2\min\left\{\tilde{g}_m\tilde{\mu}_m,\, \gamma_{2m}\tilde{g}_{2m}\tilde{\mu}_m^d\right\}\right]\|x-z\|,$$

which after taking the infimum over all z establishes Theorem 5.3.3. □

Remark 5.3.5. As we can observe in the above theorem, we give two possible upper bounds for the parameter $\mathbf{L}_m^{\mathrm{ch}}$ using the super-democracy and disjoint-super-democracy parameters. The reason to distinguish these estimates is because, in general, for $m \geq 1$,

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (\tilde{\mu}_m^d)^2 \tag{5.24}$$

and

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq \gamma_m(\tilde{\mu}_m^d + 2\kappa), \tag{5.25}$$

where $\kappa = 1$ or 2 depending if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Indeed, the left inequality in (5.24) is immediate by definition, and the right one follows from

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} = \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_C\|}\,\frac{\|\mathbf{1}_C\|}{\|\mathbf{1}_{\varepsilon' B}\|} \leq (\tilde{\mu}_m^d)^2,$$

for any $|A| = |B| \leq m$ and any C disjoint with $A\cup B$ with $|C| = |A|$. Concerning the right inequality in (5.25), we use that if $|A| \leq |B| \leq m$ then

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\| + \|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\|}{\|\mathbf{1}_{\eta(B\setminus A)}\|} + \frac{\|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m\, \tilde{\mu}_m^d + 2\kappa\gamma_m,$$

using in the last step Lemma 4.0.6. From (5.25) we see that $\tilde{\mu}_m \approx \tilde{\mu}_m^d$ when $\mathscr{B}$ is unconditional for constant coefficients. Also, the same property is valid for general Schauder bases without the necessity to assume that the basis is unconditional for constant coefficients (to prove that, we only have to repeat the argument of Theorem 2.2.11). Finally, the squared bound of 5.24 is essentially optimal as we will show in Example 5.3.7.

Before to show the Example 5.3.7, we give the following lemma.

Lemma 5.3.6.

$$\tilde{\mu}_m^d = \sup\left\{\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} : |B| = |A| \leq m,\ A\cap B = \emptyset,\ \varepsilon\in\Psi_A, \eta\in\Psi_B\right\} =: \mathfrak{D}_m^d. \tag{5.26}$$

Proof. Since $\mathfrak{D}_m^d \leq \tilde{\mu}_m^d$, we only have to show that $\mathfrak{D}_m^d \geq \tilde{\mu}_m^d$. Let $\varepsilon\in\Psi_A$, $\eta\in\Psi_B$ and $|B| \leq |A| \leq m$ with $A\cap B = \emptyset$. Pick any set C disjoint with $A\cup B$ such that $|B| + |C| = |A|$. We now use the elementary inequality

$$\|x\| = \left\|\frac{x+y}{2} + \frac{x-y}{2}\right\| \le \max\{\|x+y\|, \|x-y\|\}, \tag{5.27}$$

with $x = \mathbf{1}_{\eta B}$ and $y = \mathbf{1}_C$. Let $|\eta'| = 1$ be such that $\eta'|_B = \eta|_B$ and $\eta'|_C = \pm 1$, according to the sign that reaches the maximum in (5.27). Then $\|\mathbf{1}_{\eta B}\| \le \|\mathbf{1}_{\eta'(B\cup C)}\| \le \mathfrak{D}_m^d \|\mathbf{1}_{\varepsilon A}\|$, and the result follows. □

Example 5.3.7. There exists a Banach space $\mathbb{X}$ with an SM-basis $\mathscr{B}$ such that

$$\limsup_{m\to\infty} \frac{\tilde{\mu}_m}{[\tilde{\mu}_m^d]^{2-\varepsilon}} = \limsup_{m\to\infty} \frac{\mu_m}{[\mu_m^d]^{2-\varepsilon}} = \infty, \quad \forall\, \varepsilon > 0.$$

Proof. Let $N_0 = 1$, and define recursively $N_k = 2^{2^{N_{k-1}}}$, and $N_k' = N_1 + \ldots + N_{k-1}$. Consider the blocks of integers

$$S_k = \left\{N_k' + 1, \ldots, N_k' + N_k\right\},$$

and denote the tail blocks by $T_k = \cup_{j\ge k+1} S_j$. Finally, let

$$\mathfrak{N}_k = \Big\{(\sigma_j)_{j\in S_k} \mid \sigma_j \in \{\pm 1\} \quad \text{and} \quad \sum_{j\in S_k} \sigma_j = 0\Big\}.$$

We define a real Banach space $\mathbb{X}$ as the closure of c_{00} with the norm

$$\|x\| \;=\; \max\Big\{\; \|x\|_\infty,\ \sup_{k\ge 1} \alpha_k \sup_{\sigma\in\mathfrak{N}_k} \big|\langle \mathbf{1}_{\sigma S_k}, x\rangle\big|,\ \sup_{k\ge 1} \beta_k \sup_{\substack{S\subset T_k\\ |S|=N_k}} \sum_{j\in S} |x_j| \;\Big\},$$

where the weights α_k and β_k are chosen as follows:

$$\alpha_k = 2^{-N_{k-1}} = \frac{1}{\log_2(N_k)} \quad \text{and} \quad \beta_k = \frac{1}{\sqrt{N_k}}.$$

Observe that

$$N_k' = N_1 + \ldots + N_{k-1} \le 2N_{k-1} = 2\log_2\log_2(N_k) \quad \text{and} \quad \frac{\alpha_k}{\beta_k} = \frac{\sqrt{N_k}}{\log_2(N_k)}.$$

Claim 1: $\quad \tilde{\mu}_{N_k} \ge \mu_{N_k} \ge \dfrac{N_k/2}{(\log_2(N_k))\sqrt{\log_2\log_2(N_k)}}$, for all $k \ge 1$.

Proof. Pick any $A \subset S_k \cup S_{k+1}$ such that $|A| = N_k$ and $|A \cap S_k| = |A \cap S_{k+1}| = N_k/2$. Then

$$\|\mathbf{1}_A\| \ge \alpha_k N_k/2 = \frac{N_k/2}{\log_2(N_k)}.$$

Next, pick $B = S_k$, so that $|B| = |A| = N_k$ and

$$\|\mathbf{1}_B\| = \max\Big\{\; 1,\ \alpha_k \cdot 0,\ \sup_{n\le k-1} \beta_n N_n \;\Big\} = \beta_{k-1} N_{k-1} = \sqrt{N_{k-1}} = \sqrt{\log_2\log_2(N_k)}.$$

Then $\mu_{N_k} \ge \|\mathbf{1}_A\|/\|\mathbf{1}_B\| \ge \frac{N_k/2}{(\log_2(N_k))\sqrt{\log_2\log_2(N_k)}}$. □

Claim 2: $\mu^d_{N_k} \le \tilde{\mu}^d_{N_k} \le \sqrt{N_k}$, for all $k \ge 2$.

Proof. Let A, B be any pair of disjoint sets with $|A| = |B| \le N_k$, and let $|\varepsilon| = |\eta| = 1$. It is enough to consider the condition of $|A| = |B|$ using Lemma 5.3.6. If $|A| = |B| \le \sqrt{N_k}$, then the trivial bounds $\|\mathbf{1}_{\varepsilon A}\| \le |A|$ and $\|\mathbf{1}_{\eta B}\| \ge 1$ give

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \le \sqrt{N_k}.$$

So, it remains to consider the cases $\sqrt{N_k} < |A| = |B| \le N_k$. We split A into three parts

$$A_0 = A \cap S_k, \quad A_+ = A \cap T_k, \quad A_- = A \cap [S_1 \cup \ldots \cup S_{k-1}].$$

Then, we have the following upper bound

$$\begin{aligned} \|\mathbf{1}_{\varepsilon A}\| &\le \max\Big\{1, \sup_{n<k} \alpha_n |A_-|, \ \alpha_k |A_0|, \sup_{n>k} \alpha_n N_k, \sup_{n<k} \beta_n N_n, \sup_{n\ge k} \beta_n |A| \Big\} \\ &\le \max\Big\{ N'_k, \ \alpha_k |A_0|, \ \beta_k |A| \Big\}, \end{aligned}$$

due to the elementary inequalities

- $\sup_{n<k} \alpha_n |A_-| \le |A_-| \le N'_k$
- $\sup_{n>k} \alpha_n N_k = \alpha_{k+1} N_k = N_k 2^{-N_k} \le 1$
- $\sup_{n<k} \beta_n N_n = \sqrt{N_{k-1}} \le N_{k-1} \le N'_k$
- $\sup_{n\ge k} \beta_n |A| = \beta_k |A|$.

Moreover, since $\beta_k |A| \le \min\{\beta_k N_k = \sqrt{N_k}, \ \alpha_k |A| \}$, we derive

$$\|\mathbf{1}_{\varepsilon A}\| \le \max\{\sqrt{N_k}, \alpha_k |A_0|\} \quad \text{and} \quad \|\mathbf{1}_{\varepsilon A}\| \le \max\{N'_k, \alpha_k |A|\}. \tag{5.28}$$

We now give a lower bound for $\|\mathbf{1}_{\eta B}\|$. The key estimate will rely on the following

Lemma 5.3.8. Let $B_0 = B \cap S_k$ and $B_0^c = S_k \setminus B_0$. Then

$$\sup_{\sigma \in \mathfrak{N}_k} \big| \langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0} \rangle \big| \ge \min\{|B_0|, |B_0^c|\}. \tag{5.29}$$

Proof. If $|B_0| \le N_k/2$, then we may select any $\sigma \in \mathfrak{N}_k$ such that $\sigma|_{B_0} = \eta$ (which is possible since $|B_0^c| \ge |B_0|$), which gives

$$|\langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0} \rangle| = |B_0| = \min\{|B_0|, |B_0^c|\}.$$

Assume now that $|B_0| > N_k/2$. Pick any $S \subset B_0$ with $|S| = |B_0^c| = N_k - |B_0|$. Choose $\nu \in \{-1,1\}^{B_0^c}$ so that $\sum_{i \in S} \eta_i + \sum_{i \in B_0^c} \nu_i = 0$. Choose $\tau \in \{-1,1\}^{B_0 \setminus S}$ so that $\sum_{i \in B_0 \setminus S} \tau_i = 0$. Replacing τ by $-\tau$, if necessary, we may assume that $\sum_{i \in B_0 \setminus S} \tau_i \eta_i \ge 0$. Finally, define $\sigma \in \mathfrak{N}_k$ by setting

$$\sigma|_S = \eta|_S, \quad \sigma|_{B_0^c} = \nu|_{B_0^c}, \quad \sigma|_{B_0 \setminus S} = \tau|_{B_0 \setminus S}.$$

Then,

$$|\langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0}\rangle| = \sum_{i\in S} \eta_i^2 + \sum_{i\in B_0\setminus S} \tau_i \eta_i \ge |S| = |B_0^c| = \min\{|B_0|, |B_0^c|\}. \qquad \square$$

From the lemma and the definition of the norm we see that

$$\|\mathbf{1}_{\eta B}\| \ge \max\Big\{ 1,\ \alpha_k \min\{|B_0|, |B_0^c|\},\ \beta_k |B_+| \Big\}. \tag{5.30}$$

We shall finally combine the estimates in (5.28) and (5.30) to establish Claim 2. We distinguish two cases

Case 1: $\min\{|B_0|, |B_0^c|\} = |B_0^c|$. Then, since $A_0 \subset B_0^c$, we see that

$$\alpha_k |A_0| \le \alpha_k |B_0^c| \le \|\mathbf{1}_{\eta B}\|,$$

and therefore the first estimate in (5.28) gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \le \frac{\max\{\sqrt{N_k}, \|\mathbf{1}_{\eta B}\|\}}{\|\mathbf{1}_{\eta B}\|} \le \sqrt{N_k}.$$

Case 2: $\min\{|B_0|, |B_0^c|\} = |B_0|$. Then, (5.30) reduces to

$$\|\mathbf{1}_{\eta B}\| \ge \max\{\ \alpha_k |B_0|,\ \beta_k |B_+|\ \} \ge \beta_k \frac{|B_0| + |B_+|}{2} = \beta_k \frac{|B| - |B_-|}{2} \ge \beta_k |B|/4,$$

since $|B_-| \le N_k' \le \sqrt{N_k}/2 \le |B|/2$, if $k \ge 2$. Also, the second bound in (5.28) reads

$$\|\mathbf{1}_{\varepsilon A}\| \le \alpha_k |A|,$$

since $N_k' \le \sqrt{N_k}/\log_2 N_k = \alpha_k \sqrt{N_k} \le \alpha_k |A|$, if $k \ge 2$. Thus

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \le \frac{\alpha_k |A|}{\beta_k |B|/4} = \frac{4\alpha_k}{\beta_k} = \frac{4\sqrt{N_k}}{\log_2(N_k)} \le \sqrt{N_k}.$$

This establishes Claim 2. $\square$

From Claims 1 and 2 we now deduce that

$$\frac{\mu_{N_k}}{[\tilde{\mu}^d_{N_k}]^{2-\varepsilon}} \ge \frac{N_k^{\varepsilon/2}/2}{(\log_2(N_k))\sqrt{\log_2 \log_2(N_k)}} \to \infty,$$

and therefore

$$\limsup_{N\to\infty} \frac{\mu_N}{[\mu^d_N]^{2-\varepsilon}} = \limsup_{N\to\infty} \frac{\tilde{\mu}_N}{[\tilde{\mu}^d_N]^{2-\varepsilon}} = \infty. \qquad \square$$

5.3.2 Lower bounds of $\mathbf{L}_m^{\mathrm{ch}}$

In this section, we introduce the *de la Vallée-Poussin* type operators. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. Recall that $P_m(x) = \sum_{n=1}^m \mathbf{e}_n^*(x)\mathbf{e}_n$ and

$$F_N(x) = \frac{1}{N}\sum_{n=1}^N P_n(x) = \sum_{n=1}^N \big(1 - \frac{n-1}{N}\big)\mathbf{e}_n^*(x)\mathbf{e}_n.$$

For $M > N$ we define the *de la Vallée-Poussin type operators* as

$$\begin{aligned} V_{N,M}(x) &= \frac{M}{M-N}F_M(x) - \frac{N}{M-N}F_N(x) \\ &= \sum_{n=1}^N \mathbf{e}_n^*(x)\mathbf{e}_n + \sum_{n=N+1}^M \big(1 - \frac{n-N-1}{M-N}\big)\,\mathbf{e}_n^*(x)\mathbf{e}_n. \end{aligned} \tag{5.31}$$

In particular, observe that, for β as in (5.2) we have

$$\max\big\{\|V_{N,M}\|, \|I - V_{N,M}\|\big\} \le \frac{M+N}{M-N}\beta. \tag{5.32}$$

Theorem 5.3.9. If $\mathscr{B}$ is an SM-Cesàro basis in $\mathbb{X}$ with constant β, then for every $c \ge 2$

$$\mathbf{L}_m^{\mathrm{ch}} \ge \frac{1}{\beta^2}\frac{c-1}{c+1}\vartheta_{m,c}, \quad \forall\, m \in \mathbb{N},$$

where $\vartheta_{m,c}$ is defined in 5.14.

Before the proof of Theorem 5.3.9, we show the following lemma.

Lemma 5.3.10. Let $|A| = |B| \le m$ and let $y \in \mathbb{X}$ be such that $|y|_\infty \le 1$ and $A > c(B \cup \operatorname{supp} y)$ with $B \cap \operatorname{supp}(y) = \emptyset$. Then

$$\mathbf{L}_m^{\mathrm{ch}} \ge \frac{1}{\beta^2}\frac{c-1}{c+1}\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B} + y\|}, \quad \forall \varepsilon \in \Upsilon_A, \eta \in \Upsilon_B. \tag{5.33}$$

Proof. Pick a large integer $\lambda > 1$ and a set $C > \lambda A$ such that $|B \cup C| = m$. Let

$$x = \mathbf{1}_{\varepsilon A} + y + \mathbf{1}_{\eta B} + \mathbf{1}_C.$$

$B \cup C \in \mathscr{G}(x,m)$, and hence for some Chebyshev greedy operator we have

$$x - \mathfrak{CG}_m(x) = \mathbf{1}_{\varepsilon A} + y + \sum_{n \in B \cup C} a_n \mathbf{e}_n,$$

for suitable scalars $a_n \in \mathbb{F}$. Choosing $\mathbf{1}_{\varepsilon A} + \mathbf{1}_C$ as m-term approximant of x we see that

$$\|x - \mathfrak{CG}_m(x)\| \le \mathbf{L}_m^{\mathrm{ch}}\sigma_m(x) \le \mathbf{L}_m^{\mathrm{ch}}\,\|\mathbf{1}_{\eta B} + y\|.$$

On the other hand, calling $N = \max(B \cup \operatorname{supp} y)$ and $L = \max A$ we have

$$(\mathrm{Id}_{\mathbb{X}} - V_{N,cN}) \circ V_{L,\lambda L}\big(x - \mathfrak{CG}_m x\big) = \mathbf{1}_{\varepsilon A}$$

Thus,
$$\|x-\mathfrak{CG}_m(x)\| \geq \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathrm{Id}_{\mathbb{X}}-V_{N,cN}\|\|V_{L,\lambda L}\|} \geq \frac{c-1}{(c+1)\beta}\,\frac{\lambda-1}{(\lambda+1)\beta}\,\|\mathbf{1}_{\varepsilon A}\|.$$
Therefore we obtain
$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{\beta^2}\,\frac{c-1}{c+1}\,\frac{\lambda-1}{\lambda+1}\,\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}+y\|}$$
which letting $\lambda\to\infty$ yields (5.33). □

Proof of Theorem 5.3.9: We prove first that, if $c\geq 2$, then for all $A,B\subset\mathbb{N}$ such that $B>cA$ with $|A|=|B|\leq m$ it holds

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{\beta}\,\frac{c-1}{c+1}\,\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\varepsilon\in\Psi_A,\eta\in\Psi_B. \tag{5.34}$$

Pick any set $C>B$ such that $|B\cup C|=m$, and let

$$x=\mathbf{1}_{\varepsilon A}+\mathbf{1}_{\eta B}+\mathbf{1}_C.$$

Then $B\cup C\in\mathscr{G}(x,m)$, and hence there is a Chebyshev greedy operator so that

$$x-\mathfrak{CG}_m(x)=\mathbf{1}_{\varepsilon A}+\sum_{n\in B\cup C}a_n\mathbf{e}_n,$$

for some scalars $a_n\in\mathbb{F}$. Clearly,

$$\|x-\mathfrak{CG}_m(x)\|\leq \mathbf{L}_m^{\mathrm{ch}}\sigma_m(x)\ \leq\ \mathbf{L}_m^{\mathrm{ch}}\,\|\mathbf{1}_{\eta B}\|,$$

using $z=\mathbf{1}_{\varepsilon A}+\mathbf{1}_C$ an m-term approximant. On the other hand, let $N=\max A$. Since $\min B\cup C>cN$, then (5.31) yields
$$V_{N,cN}(x-\mathfrak{CG}_m x)=\mathbf{1}_{\varepsilon A}.$$
Therefore, (5.32) implies that

$$\|x-\mathfrak{CG}_m(x)\|\geq\frac{\|V_{N,cN}(x-\mathfrak{CG}_m x)\|}{\|V_{N,cN}\|}\geq\frac{c-1}{(c+1)\beta}\,\|\mathbf{1}_{\varepsilon A}\|.$$

We have therefore proved (5.34).

Now, using Lemma 5.3.10 with $y=0$, we have that for all sets $|A|=|B|\leq m$ satisfy $A>cB$ then
$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{\beta^2}\,\frac{c-1}{c+1}\,\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\varepsilon\in\Psi_A,\eta\in\Psi_B. \tag{5.35}$$
This together with (5.34) is enough to establish Theorem 5.3.9. □

Remark 5.3.11. When $\mathscr{B}$ is Schauder, a similar proof gives the following lower bound, which is also obtained in [71, Theorem 2.2]:

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{(\mathfrak{K}_b+1)}\sup\Big\{\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} : |A|=|B|=m,\ A>B \text{ or } B>A\ ,\ \varepsilon\in\Psi_A,\eta\in\Psi_B\Big\}.$$

The statement for Cesàro bases, however, will be needed for the applications in the subsection 5.3.3.

Finally, we prove a similar lower bound but in the context of ρ-admissible bases.

Proposition 5.3.12. Let $\mathscr{B}$ be an SM-ρ-admissible basis. Then

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{\vartheta_m}{(\rho+1)}, \quad \forall\, m \in \mathbb{N}. \tag{5.36}$$

Proof. Fix $A \subset \mathbb{N}$ such that $|A| \leq m$. Choose C disjoint with A such that $|A \cup C| = m$. Let $n_0 = n_0(A \cup C)$ as in the definition of ρ-admissibility, which we may assume larger than $\max A \cup C$. Pick any B with $\min B \geq n_0$ and $|B| = |A|$, and any $\varepsilon \in \Psi_A, \eta \in \Psi_B$. Let $x := \mathbf{1}_{\varepsilon A} + \mathbf{1}_C + \mathbf{1}_{\eta B}$. Then $A \cup C \in \mathscr{G}(x,m)$, and there is a Chebyshev greedy operator with $\mathfrak{CG}_m(x)$ supported in $A \cup C$. Thus,

$$\|x - \mathfrak{CG}_m(x)\| \leq \mathbf{L}_m^{\mathrm{ch}}\, \sigma_m(x) \leq \mathbf{L}_m^{\mathrm{ch}}\, \|x - (\mathbf{1}_{\eta B} + \mathbf{1}_C)\| = \mathbf{L}_m^{\mathrm{ch}}\, \|\mathbf{1}_{\varepsilon A}\|.$$

On the other hand, using the definition of ρ-admissibility (see Definition (5.1.1)) one obtains

$$\|x - \mathfrak{CG}_m(x)\| \geq \frac{\|\mathbf{1}_{\eta B}\|}{\rho+1}.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{(\rho+1)} \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}.$$

We now assume additionally that $\min B \geq n_0 + m$, and pick $D \subset [n_0, n_0 + m - 1]$ such that $|B| + |D| = m$. Let $y = \mathbf{1}_{\varepsilon A} + \mathbf{1}_{\eta B} + \mathbf{1}_D$. Then $B \cup D \in \mathscr{G}(y,m)$ and a similar reasoning gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\rho} \leq \|y - \mathfrak{CG}_m(y)\| \leq \mathbf{L}_m^{\mathrm{ch}}\, \sigma_m(y) \leq \mathbf{L}_m^{\mathrm{ch}}\, \|\mathbf{1}_{\eta B}\|.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{1}{\rho+1} \max\left\{ \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}, \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \right\},$$

and taking the supremum over all $|B| = |A|$ with $B \geq (n_0 + m)A$ and all $\varepsilon \in \Psi_A, \eta \in \Psi_B$, we see that

$$\mathbf{L}_m^{\mathrm{ch}} \geq \frac{\vartheta_{n_0+m}(A)}{\rho+1} \geq \frac{\inf_{c \geq 1} \vartheta_c(A)}{\rho+1}.$$

Finally, a supremum over all $|A| \leq m$ leads to (5.36). □

5.3.3 The trigonometric system in $L_p(\mathbb{T})$

Consider $\mathscr{B} = \{e^{inx}\}_{n \in \mathbb{Z}}$ in $L_p(\mathbb{T})$ for $1 \leq p < \infty$, and in $C(\mathbb{T})$ if $p = \infty$ as in the Example 4.3.3.

As we have said at the beginning of Section 4.2 of Chapter 4, in [76], Temlyakov showed that

$$c_p m^{|\frac{1}{p} - \frac{1}{2}|} \leq \mathbf{L}_m \leq 1 + 3m^{|\frac{1}{p} - \frac{1}{2}|},$$

for some $c_p > 0$ and all $1 \leq p \leq \infty$. We show that for the TCGA, the behavior of $\mathbf{L}_m^{\mathrm{ch}}$ is the same.

Proposition 5.3.13. Let $1 \le p \le \infty$. Then there exists $c_p > 0$ such that

$$\mathbf{L}_m^{\mathrm{ch}} \ge c_p m^{|\frac{1}{p}-\frac{1}{2}|}, \quad \forall\, m \in \mathbb{N}. \tag{5.37}$$

Remark 5.3.14. In [71, Theorem 2.1], Shao and Ye have established that for $1 < p \le \infty$,

$$\mathbf{L}_m^{\mathrm{ch}} \approx m^{|\frac{1}{p}-\frac{1}{2}|}. \tag{5.38}$$

They do not establish the case $p = 1$ and their the proof of the case $p = \infty$ seems to contain some gaps and may not be complete.

We remark that in the cases $p = 1$ and $p = \infty$ the trigonometric system is not a Schauder basis, but it is a Cesàro basis. So we may use the lower bounds in Theorem 5.3.9, namely

$$\mathbf{L}_m^{\mathrm{ch}} \ge c'_p \sup_{\substack{|A|=|B|\le m \\ A>2B \text{ or } B>2A}} \sup_{\varepsilon\in\Psi_A,\eta\in\Psi_B} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}. \tag{5.39}$$

- Case $1 < p \le 2$. Assume that $m = 2\ell+1$ or $2\ell+2$ (that is, $\ell = \lfloor \frac{m-1}{2} \rfloor$). We choose $B = \{-\ell, ..., \ell\}$, so that $\mathbf{1}_B = D_\ell$ is the ℓ-th Dirichlet kernel, and hence

  $$\|\mathbf{1}_B\|_p = \|D_\ell\|_p \approx m^{1-\frac{1}{p}}.$$

  Next we take a lacunary set $A = \{2^j : j_0 \le j \le j_0 + 2\ell\}$, so that

  $$\|\mathbf{1}_A\|_p \approx \sqrt{m}, \tag{5.40}$$

  and where j_0 is chosen such that $2^{j_0} \ge m$, and hence $A > 2B$. Then, (5.39) implies

  $$\mathbf{L}_m^{\mathrm{ch}} \ge c_p \frac{m^{1/2}}{m^{1-\frac{1}{p}}} = c_p m^{|\frac{1}{p}-\frac{1}{2}|}.$$

- Case $2 \le p < \infty$. The same proof works in this case, just reversing the roles of A and B.

- Case $p = \infty$. We replace the lacunary set by a Rudin-Shapiroo polynomial of the form

  $$R(x) = e^{iNx} \sum_{n=0}^{2^L-1} \varepsilon_n e^{inx}, \quad \text{with } \varepsilon_n \in \{\pm 1\},$$

  where L is such that $2^L \le m < 2^{L+1}$ (see for instance [57, p. 33]). Then, $R = \mathbf{1}_{\varepsilon B}$ with $B = N + \{0, 1, \ldots, 2^L - 1\}$ and

  $$\|\mathbf{1}_{\varepsilon B}\|_\infty = \|R\|_\infty \approx \sqrt{m}.$$

  If we pick $N \ge 2 \cdot 2^L$, then $B > 2A$ with $A = \{\pm 1, \ldots, \pm(2^L - 1)\}$. Finally,

  $$\|\mathbf{1}_A\|_\infty = \|D_{2^L-1} - 1\|_\infty \approx m.$$

  So, (5.39) implies the desired bound.

- Case $p=1$. We use the lower bound in Lemma 5.3.10, namely

$$\mathbf{L}_m^{\mathrm{ch}} \geq c_1' \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B+y\|}, \tag{5.41}$$

for all $|A|=|B|\leq m$ and all y such that $A>2(B\cup \operatorname{supp}(y))$ with $B\cap \operatorname{supp}(y)=\emptyset$ and $|y|_\infty \leq 1$. As before, let $m=2\ell+1$ or $2\ell+2$, and choose the same sets A and B as in the case $1<p\leq 2$. Next choose y so that the vector

$$V_\ell = \mathbf{1}_B + y,$$

is a de la Vallée-Poussin kernel as in [57, p. 15]. Then, the Fourier coefficients $\mathbf{e}_n^*(y)$ have modulus less tan or equal to 1 and are supported in $\{n : \ell < |n| \leq 2\ell+1\}$, so the condition $A>2(B\cup \operatorname{supp} y)$ with $B\cap \operatorname{supp}(y)=\emptyset$ holds if $2^{j_0}\geq 2m+1$. Finally,

$$\|\mathbf{1}_B+y\|_1 = \|V_\ell\|_1 \leq 3,$$

so the bound $\mathbf{L}_m^{\mathrm{ch}} \gtrsim \sqrt{m}$ follows from (5.41).

Remark 5.3.15. Using the trivial upper bound $\mathbf{L}_m^{\mathrm{ch}} \leq \mathbf{L}_m \lesssim m^{|\frac{1}{p}-\frac{1}{2}|}$, we conclude that $\mathbf{L}_m^{\mathrm{ch}} \approx m^{|\frac{1}{p}-\frac{1}{2}|}$ for all $1\leq p\leq \infty$.

5.4 Open questions

- In the characterization of semi-greedy bases, we have used the ρ-admissibility condition. In the characterization of greedy, almost-greedy and quasi-greedy bases it is enough to work with general SM-bases.

  Question 1: is it possible to remove the condition ρ-admissibility in the characterization of semi-greediness and characterize them for general SM-bases?

- In relation to the almost-greedy bases with constant $C_{al}=1$, we propose the following question.

  Question 2: is it possible to prove that 1-semi-greediness is equivalent to 1-almost-greediness? Equivalently, if $\mathscr{B}$ is 1-semi-greedy, is $\mathscr{B}$ 1-symmetric for largest coefficients?

Chapter 6

Weight-greedy-type bases

In 2000, A. Cohen, R. A. DeVore and R. Hochmuth (see [29]) considered the following: let $(\phi_I)_I$ be a wavelet basis indexed by dyadic intervals $I \subset \mathscr{D}$, where $\mathscr{D}$ is the collection of all dyadic intervals. Take $\alpha \in (-\infty, 1)$ and assign to each index set $\Lambda \in \mathscr{D}$ the measure

$$w_\alpha(\Lambda) := \sum_{I \in \Lambda} |I|^\alpha.$$

Then, in the Hardy space H_p, $0 < p < \infty$, the authors study the following error: given $f \in H_p$,

$$\inf_{S \in \Sigma_t} \|f - S\|_{H_p}, \tag{6.1}$$

where Σ_t is the set of the linear combinations generated by ϕ_Λ with $\Lambda \in \mathscr{D}$ such that $w_\alpha(\Lambda) \leq t$. In 2013, similar measures where used by E. Hernández and D. Vera to prove some inclusions of approximation spaces (see [50]).

With this idea, in the year 2006, G. Kerkyacharian, D. Picard and V. N. Temlyakov (see [58]) defined an extension of greedy bases, the so called w-greedy-type bases, and in this chapter we study this generalization and some characterizations. For that, we need the following definitions: let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis in a Banach space $\mathbb{X}$. We consider a weight $w = (w_i)_{i=1}^\infty \in (0,\infty)^{\mathbb{N}}$ and for $A \subset \mathbb{N}$, we denote the corresponding measure by

$$w(A) = \sum_{i \in A} w_i.$$

For each $\delta > 0$, we consider the following extensions of the error of approximation (6.1):

$$\sigma_\delta^w(x, \mathscr{B})_{\mathbb{X}} := \sigma_\delta^w(x) = \left\{ \left\| x - \sum_{i \in A} a_i \mathbf{e}_i \right\| : a_i \in \mathbb{F}, A \in \mathbb{N}^{<\infty}, w(A) \leq \delta \right\},$$

and

$$\widetilde{\sigma}_\delta^w(x, \mathscr{B})_{\mathbb{X}} := \widetilde{\sigma}_\delta^w(x) = \left\{ \|x - P_A(x)\| : A \in \mathbb{N}^{<\infty}, w(A) \leq \delta \right\}.$$

We remind some notation presented in the above chapters: $\mathbb{N}^{<\infty}$ is the collection of finite sets, $A < B$ means that $\max A < \min B$ and $|x|_\infty := \sup_{j \in \operatorname{supp}(x)} |\mathbf{e}_j^*(x)|$ for $x \in \mathbb{X}$. The results that we give in this chapter are in [21].

6.1 w-democracy-like properties

In the following sections, we will need suitable notions of w-democracy to characterize w-greedy-type bases. We present now the definitions and we will study some properties and relations between them.

As in Chapter 2, we define the set $\mathfrak{F}(w)$, where w is a weight, as the family of all 5-tuples $(A,B,\varepsilon,\varepsilon',x)$ such that

$$A,B \in \mathbb{N}^{<\infty}, \quad w(A) \leq w(B), \quad |x|_\infty \leq 1, \quad \operatorname{supp}(x) \cap (A \cup B) = \emptyset, \quad \varepsilon \in \Psi_A, \quad \varepsilon' \in \Psi_B.$$

Consider the following subsets:

- $\mathfrak{F}_d(w)$ is the subset of $\mathfrak{F}(w)$ where $A \cap B = \emptyset$.
- $\mathfrak{F}_c(w)$ is the subset of $\mathfrak{F}_d(w)$ where $A < B$.

Finally, we write $\mathfrak{F}'(w)$ for the subset of $\mathfrak{F}(w)$ where $x = 0$ and likewise $\mathfrak{F}'_d(w)$ and $\mathfrak{F}'_c(w)$. The next table shows all w-democracy-like properties that we need in this chapter.

Name	**Inequality**	
w-democracy	$\|\mathbf{1}_A\| \leq C_d^w \|\mathbf{1}_B\| \quad \forall (A,B,1,1) \in \mathfrak{F}'(w)$	(6.2)
w-super-democracy	$\|\mathbf{1}_{\varepsilon A}\| \leq C_s^w \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'(w)$	(6.3)
w-disjoint-super-democracy	$\|\mathbf{1}_{\varepsilon A}\| \leq C_{sd}^w \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'_d(w)$	(6.4)
w-conservative	$\|\mathbf{1}_A\| \leq C_c^w \|\mathbf{1}_B\| \quad \forall (A,B,1,1) \in \mathfrak{F}'_c(w)$	(6.5)
w-super-conservative	$\|\mathbf{1}_{\varepsilon A}\| \leq C_{sc}^w \|\mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon') \in \mathfrak{F}'_c(w)$	(6.6)
w-SLC	$\|x + \mathbf{1}_{\varepsilon A}\| \leq C_a^w \|x + \mathbf{1}_{\varepsilon' B}\| \quad \forall (A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d(w)$	(6.7)

The constants that appear in the above table are the least constant that verifies the respective conditions and we will say that $\mathscr{B}$ is C_d^w-w-democratic if (6.2) holds, C_s^w-w-super-democratic if (6.3) holds, etc.

Remark 6.1.1. By definition, if $\mathscr{B}$ is w-super-democratic, then it is w-disjoint-super-democratic, w-democratic, w-super-conservative and w-conservative with

$$C_c^w \leq C_d^w \leq C_s^w \text{ and } C_c^w \leq C_{sc}^w \leq C_{sd}^w \leq C_s^w.$$

The following proposition is an extension of Proposition 2.2.8 and can be found in [21] and [22].

Proposition 6.1.2. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-basis in a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_a^w-w-SLC, then $\mathscr{B}$ is w-disjoint-super-democratic and w-super-democratic with

$$C_{sd}^w \leq C_a^w \text{ and } C_s^w \leq (2\kappa)C_a^w,$$

where $\kappa = 1$ if $\mathbb{F} = \mathbb{R}$ and $\kappa = 2$ if $\mathbb{F} = \mathbb{C}$.

b) $\mathscr{B}$ is C_a^w-w-SLC if and only if for any $\varepsilon' \in \Psi_B$, $x \in \mathbb{X}$, $B \cap \operatorname{supp}(x) = \emptyset$, $w(A) \leq w(B)$ and $1 \geq |x|_\infty$, there exists a constant C_1 such that

$$\|x\| \leq C_1 \|x - P_A(x) + \mathbf{1}_{\varepsilon' B}\|. \tag{6.8}$$

Also, $C_1 = C_a^w$.

Proof. First, we prove a). Of course, (6.7) implies (6.4) with $C_a^w \geq C_{sd}^w$. Assume that $\mathscr{B}$ is C_a^w-w-SLC, take $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}'(w)$ and show (6.3). First of all, using Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D \subseteq A} \|\mathbf{1}_{\varepsilon' D}\|. \tag{6.9}$$

Now, with the decomposition $\mathbf{1}_{\varepsilon' D} = \mathbf{1}_{\varepsilon'(D \setminus B)} + \mathbf{1}_{\varepsilon'(D \cap B)}$, since $w(A) \leq w(B)$, then $w(D \setminus B) \leq w(B \setminus D)$. Using the w-SLC,

$$\|\mathbf{1}_{\varepsilon' D}\| \leq C_a^w \|\mathbf{1}_{\varepsilon'(B \setminus D)} + \mathbf{1}_{\varepsilon'(D \cap B)}\| = C_a^w \|\mathbf{1}_{\varepsilon' B}\|. \tag{6.10}$$

Using (6.10) in (6.9), we obtain the result.

Finally, we show b). Assume that $\mathscr{B}$ is C_a^w-w-SLC. Take A, B, x, ε' as in the statement. From (6.7), for any $\varepsilon \in \Psi_A$,

$$\|P_{A^c}(x) + \mathbf{1}_{\varepsilon A}\| \leq C_a^w \|P_{A^c}(x) + \mathbf{1}_{\varepsilon' B}\|.$$

Now, by Corollary 1.3.2, we obtain the first part of the result. Now, assume that we have (6.8) and take $(A, B, \varepsilon, \varepsilon', x) \in \mathfrak{F}_d(w)$ to show (6.7). If we define $x' := x + \mathbf{1}_{\varepsilon A}$ and we apply (6.8) for x', we obtain the result. □

Definition 6.1.3. Let $v = (v_n)_{n=1}^\infty$ and $w = (w_n)_{n=1}^\infty$ be weights. We say that v is *equivalent* to w, written $v \approx w$, whenever there exist positive real constants $0 < a \leq b < \infty$ satisfying

$$av_n \leq w_n \leq bv_n \quad \text{for all } n \in \mathbb{N}.$$

This definition appears in the paper [36] where the authors showed that w-super-democracy and v-super-democracy are equivalent for $w \approx v$. Here, we extend this result for the symmetry for largest coefficients. The following proposition can be found in [21].

Proposition 6.1.4. Let v,w weights and suppose that $v \approx w$. Then every SM-basis $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$ with the w-SLC is also v-SLC.

Proof. Take $(A,B,\varepsilon,\varepsilon',x) \in \mathfrak{F}_d(v)$. Invoking Lemma (2.2.6), it is enough to consider $x \in \mathbb{X}$ with $|\operatorname{supp}(x)| < \infty$. We set

$$\Gamma = \{n \in A : w_n \geq w(B)\}.$$

Observe that

$$w(A) \leq b \cdot v(A) \leq b \cdot v(B) \leq \frac{b}{a} \cdot w(B),$$

which gives us

$$w(A) \geq w(\Gamma) \geq |\Gamma| \cdot w(B) \geq |\Gamma| \cdot \frac{a}{b} \cdot w(A),$$

and hence $|\Gamma| \leq b/a$. Next, we give the following partition of $A \setminus \Gamma$: $A_1 < \ldots < A_m$, so that for each $i = 1,\ldots,m$, the set A_i is a maximal such that $w(A_i) \leq w(B)$. Due to maximality,

$$w(B) < w(A_i) + w(A_{i+1}) \text{ for all } i = 1,\ldots,m-1.$$

Thus,

$$(m-1) \cdot w(B) < \sum_{i=1}^{m-1} [w(A_i) + w(A_{i+1})] < 2 \cdot w(A \setminus \Gamma) \leq 2 \cdot w(A) \leq \frac{2b}{a} \cdot w(B).$$

This gives us

$$m \leq \frac{2b}{a} + 1.$$

Hence, using the bounds of $|\Gamma|$, m and the condition of the w-symmetry for largest coefficients,

$$\begin{aligned}
\|x + \mathbf{1}_{\varepsilon A}\| &\leq \|\mathbf{1}_{\varepsilon\Gamma}\| + \|x + \sum_{i=1}^{m} \mathbf{1}_{\varepsilon A_i}\| \leq \sum_{n\in\Gamma} \|\mathbf{e}_n\| + \sum_{i=1}^{m} \left\| \frac{x}{m} + \mathbf{1}_{\varepsilon A_i} \right\| \\
&\leq \mathbf{c}\mathbf{c}^* |\Gamma| \|x + \mathbf{1}_{\varepsilon' B}\| + C_a^w m \|\frac{x}{m} + \mathbf{1}_{\varepsilon' B}\| \\
&\leq \frac{\mathbf{c}\mathbf{c}^* b}{a} \|x + \mathbf{1}_{\varepsilon' B}\| + C_a^w \|x + m\mathbf{1}_{\varepsilon' B}\| \\
&\leq \frac{\mathbf{c}\mathbf{c}^* b}{a} \|x + \mathbf{1}_{\varepsilon' B}\| + C_a^w m \|x + \mathbf{1}_{\varepsilon' B}\| + C_a^w (m-1) \|x\| \\
&\leq \frac{\mathbf{c}\mathbf{c}^* b}{a} \|x + \mathbf{1}_{\varepsilon' B}\| + C_a^w m \|x + \mathbf{1}_{\varepsilon' B}\| + (C_a^w)^2 (m-1) \|x + \mathbf{1}_{\varepsilon' B}\| \\
&\leq \left(\frac{\mathbf{c}\mathbf{c}^* b + (C_a^w)^2 (2b+a)}{a} \right) \|x + \mathbf{1}_{\varepsilon' B}\|.
\end{aligned}$$

□

Remark 6.1.5. In a similar fashion, one can show that, if the weights w and v are equivalent, then any w-democratic (w-superdemocratic, w-conservative, w-super-conservative) basis is also v-democratic (resp. v-superdemocratic, v-conservative, v-super-conservative).

Remark 6.1.6. The converse to Proposition 6.1.4 does not hold in general. For example, suppose the weights w,v belong to ℓ_1. By [36], the family of w-democratic (or v-democratic) bases consists precisely of those bases which are equivalent to the c_0-basis. However, w and v need not be equivalent.

Finally, for some results about the characterization of some w-greedy-type bases, we study the relation between w-super-democracy and w-disjoint-super-democracy (see [22]). For $w = (1,1,...)$, it is clear that these notions are equivalent and $C_{sd} \leq C_s \leq C_{sd}^2$. Now, we show that we have also the same relations for general weights but with different upper bound. For that, we need the following proposition that shows that under some assumptions on the weight, the basis is equivalent to the c_0-basis (see [21]).

Proposition 6.1.7. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$ and C_{sd}^w-w-disjoint-super-democratic.

i) If $A \subset \mathbb{N}$ is a finite set and $w(A) \leq \limsup_{n\to\infty} w_n$, then $\varphi_u(|A|) \leq 2\mathbf{c}C_{sd}^w$.

ii) If $\sup_n w_n = \infty$, then $\mathscr{B}$ is equivalent to the c_0-basis.

iii) If $\sum_n w_n < \infty$, then $\mathscr{B}$ is equivalent to the c_0-basis.

iv) If $\inf_n w_n = 0$, then there exist $1 < i_1 < i_2 < \dots$ so that the sequence $(\mathbf{e}_{i_k})_{k\in\mathbb{N}}$ is equivalent to the c_0-basis. Moreover, if $\lim_n w_n = 0$, then for any infinite set $A \subset \mathbb{N}$ we can select $i_1, i_2, \dots \in A$ with the properties described above.

Proof. i) Find $n_0, n_1 \in \mathbb{N}\backslash A$ so that $w_{n_0} + w_{n_1} \geq w(A)$, then $\|\mathbf{1}_{\varepsilon A}\| \leq C_{sd}^w \|\mathbf{1}_{\{n_0 \cup n_1\}}\| \leq 2\mathbf{c}C_{sd}^w$.

ii) It is enough to show that for all finite set $A \subset \mathbb{N}$, $\|\sum_{n\in A} a_n\mathbf{e}_n\| \approx \max_{n\in A}|a_n|$. On the one hand,

$$|a_n| = |\mathbf{e}_n^*\big(\sum_{n\in A} a_n\mathbf{e}_n\big)| \leq \mathbf{c}^* \|\sum_{n\in A} a_n\mathbf{e}_n\|.$$

Therefore, for any SM-basis, $\max_{n\in A}|a_n| \leq \mathbf{c}^*\|\sum_{n\in A} a_n\mathbf{e}_n\|$. On the other hand, by i), we have that $\|\mathbf{1}_{\varepsilon A}\| \leq 2\mathbf{c}C_{sd}^w$ for all choice of $\varepsilon \in \Psi_A$. Thus, by (i) of Corollary 1.3.3,

$$\|\sum_{n\in A} a_n\mathbf{e}_n\| \leq \left(\max_{n\in A}|a_n|\right) \sup_{\varepsilon\in\Psi_A} \|\mathbf{1}_{\varepsilon A}\| \leq 2\mathbf{c}C_{sd}^w.$$

iii) If $w \in \ell_1(\mathbb{N})$, $\sum_{j=2}^\infty w_j < W := \|w\|_{\ell_1(N)}$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\sum_{j=2}^\infty w_j \leq \sum_{j=1}^{n_0} w_j$. Let $A = \{2,3,...\}$, then $w(A) \leq w(\{1,...,n_0\})$. Thus, for any $\varepsilon \in \Psi_A$,

$$\|\mathbf{1}_{\varepsilon A}\| \leq \|\mathbf{1}_{\varepsilon(A\cap\{1,...,n_0\})}\| + \|\mathbf{1}_{\varepsilon(A\backslash\{1,...,n_0\})}\| \leq n_0\mathbf{c} + C_{sd}^w\|\mathbf{1}_{\varepsilon\{1,...,n_0\}}\| \leq n_0\mathbf{c}(1+C_{sd}^w), \quad (6.11)$$

Since $w(A\setminus\{1,...,n_0\}) \leq w(A) \leq w(\{1,...,n_0\})$. Finally, if $1 \in A$ and $\varepsilon \in \Psi_A$, by (6.11),

$$\|\mathbf{1}_{\varepsilon A}\| \leq \|\mathbf{e}_1\| + \|\mathbf{1}_{\varepsilon(A\setminus\{1\})}\| \leq \mathbf{c}(1 + n_0(1+C_{sd}^w)). \quad (6.12)$$

As in the proof of ii), by convexity, we obtain the result.

iv) If $\inf_n w_n = 0$, there exists a sequence $i_1 < i_2 < ....$ so that $\sum_{k=1}^\infty w_{i_k} < \infty$. Applying iii) to the sequence $\{w_{i_k}\}_k$, we obtain the first part.

Now, if $\lim_n w_n = 0$, then every infinite set $A \subset \mathbb{N}$ contains a sequence $i_1 < i_2 < ...$ with $\sum_k w_{i_k} < \infty$. Applying again iii), we conclude the result. □

Remark 6.1.8. This proposition is also valid assuming that the basis is w-symmetric for largest coefficients or w-super-democratic due to Proposition 6.1.2 and Remark 6.1.1.

Proposition 6.1.9. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) $\mathscr{B}$ is C_{sd}^w-w-disjoint-super-democratic if and only if $\mathscr{B}$ is C_s^w-w-super-democratic.

b) $\mathscr{B}$ is w-super-democratic if and only if $\mathscr{B}$ is w-democratic and unconditional for constant coefficients (see Definition 2.1.10). Quantitatively,

$$\max\{K_c, C_d^w\} \leq C_s^w \leq 4\kappa^2 C_d^w K_c,$$

where $\kappa = 1$ or 2 if $\mathbb{F}$ is the real field or the complex field, respectively.

Proof. a) Of course, (6.3) implies (6.4) with $C_{sd}^w \leq C_s^w$. Assume now that $\mathscr{B}$ is w-disjoint-super-democratic. Invoking Proposition 6.1.7, it is only necessary to assume that $\sum_{n=1}^\infty w_n = \infty$ and $\sup_n w_n < \infty$.

The following proof is based on the ideas of [36]. Take $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}'(w)$ and we need to show (6.3) assuming (6.4).

Case 1: Assume that $\limsup_{n\to\infty} w_n < w(B)$. Take $a := \max(A \cup B)$. Since $\sum_n w_n = \infty$, we can take $E = \{a+1, ..., a+N\}$ for the largest N such that $w(E) \leq w(B)$. Then,

$$w(E) \leq w(B) < w(E \cup \{a+N+1\}).$$

We denote by $n_0 := a+N+1$. In this case, since $A \cap (E \cup \{n_0\}) = \emptyset$,

$$\|\mathbf{1}_{\varepsilon A}\| \leq C_{sd}^w \|\mathbf{1}_{E\cup\{n_0\}}\| \leq C_{sd}^w \|\mathbf{1}_E\| + C_{sd}^w \mathbf{c}\mathbf{c}^* \|\mathbf{1}_{\varepsilon' B}\|. \tag{6.13}$$

On the other hand, due to $w(E) \leq w(B)$ and $E \cap B = \emptyset$,

$$\|\mathbf{1}_E\| \leq C_{sd}^w \|\mathbf{1}_{\varepsilon' B}\|. \tag{6.14}$$

Using (6.13) and (6.14), we obtain that $\mathscr{B}$ is C_s^w-w-super-democratic with $C_s^w \leq C_{sd}^w(C_{sd}^w + \mathbf{c}\mathbf{c}^*)$.

Case 2: $w(B) \leq \limsup_{n\to\infty} w_n$. Using the item i) of Proposition 6.1.7, we obtain that

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2C_{sd}^w \mathbf{c}\mathbf{c}^* \|\mathbf{1}_{\eta B}\|.$$

Now, we show b). Assume that $\mathscr{B}$ is C_s^w-w-super-democratic. It is clear by definition that $\mathscr{B}$ is K_c-unconditional for constant coefficients and C_d^w-w-democratic with $\max\{K_c, C_d^w\} \leq C_s^w$. Assume now that $\mathscr{B}$ is K_c-unconditional for constant coefficients and C_d^w-w-democratic. Take $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}'(w)$ and show (6.3). By Corollary 1.3.3,

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\kappa \sup_{D\subseteq A} \|\mathbf{1}_D\|. \tag{6.15}$$

Now, by item b) of Proposition 2.1.11 and the fact that for any subset $D \subseteq A$, $w(D) \leq w(B)$, we have that

$$\|\mathbf{1}_D\| \leq C_d^w \|\mathbf{1}_B\| \leq 2\kappa C_d^w K_c \|\mathbf{1}_{\varepsilon' B}\|. \tag{6.16}$$

Using (6.16) in (6.15), we obtain the result. □

6.2 w-greedy-type bases

In this section we study the extension of greedy-type bases introduced in Chapter 2 for the weighted case and their characterizations.

6.2.1 w-greedy bases

Definition 6.2.1 ([58])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *w-greedy* if there is a constant $C \geq 1$ such that

$$\|x - \mathscr{G}_m(x)\| \leq C\sigma^w_{w(A_m(x))}(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \tag{6.17}$$

where $A_m(x)$ is the m-th greedy set of x. The least constant C in (6.17) is denoted by $C_g^w[\mathscr{B},\mathbb{X},w] := C_g^w$ and we will say that $\mathscr{B}$ is C_g^w-*w-greedy*.

This definition was introduced in [58] and the authors gave the following characterization.

Theorem 6.2.2 ([58])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is w-greedy if and only if $\mathscr{B}$ is unconditional and w-democratic. Quantitatively:

- If $\mathscr{B}$ is C_g^w-w-greedy, $\mathscr{B}$ is unconditional and w-democratic with

$$\max\{C_d^w, K_{su}\} \leq C_g^w.$$

- If $\mathscr{B}$ is C_d^w-w-democratic and unconditional, then $\mathscr{B}$ is w-greedy with

$$C_g^w \leq K_{su} + K_{su}K_u^2 C_d^w.$$

As we have commented in Chapter 2 for $w \equiv 1$, it is not possible to recover $C_g^w = 1$ with the above theorem since $C_d^w = K_{su} = 1$ only imply that $C_g^w \leq 2$. For this reason we introduce the characterization with the w-SLC. Moreover, the example presented in Section 6.3 exhibits this fact, that is, it is an example of a basis such that $C_d^w = K_{su} = C_g^w = 1$. The following result can be found in [21] and the techniques that we use are the same than in Theorem 2.3.7.

Theorem 6.2.3. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. $\mathscr{B}$ is w-greedy if and only if $\mathscr{B}$ is unconditional and w-symmetric for largest coefficients. Quantitatively,

$$\max\{K_{su}, C_a^w\} \leq C_g^w \leq C_a^w K_{su}.$$

In particular, if $K_{su} = C_a^w = 1$, hence, the basis is w-greedy with $C_g^w = 1$.

Proof. First of all, we show the left hand inequality. Assume that $\mathscr{B}$ is C_g^w-w-greedy.

Let $x \in \mathbb{X}$ with finite support B and $A \subset B$. Define $y := P_A(x) + \sum_{n \in B \setminus A}(\alpha + \mathbf{e}_n^*(x))\mathbf{e}_n$, where

$$\alpha > \sup_{n \in A} |\mathbf{e}_n^*(x)| + \sup_{n \in B \setminus A} |\mathbf{e}_n^*(x)|.$$

The set $B\setminus A$ is the k-th greedy set for y for some $k\in\mathbb{N}$, and then

$$\|P_A(x)\| = \|y-\mathscr{G}_k(y)\| \leq C_g^w \sigma^w_{w(B\setminus A)}(y) \leq C_g^w\|y-\alpha\mathbf{1}_{B\setminus A}\| = C_g^w\|x\|.$$

Thus, by density, the basis is unconditional with $K_{su}\leq C_g^w$.

To show now the w-SLC take $(A,B,\varepsilon,\varepsilon',x)\in\mathfrak{F}_d(w)$ and show (6.7). Set $y:=x+\mathbf{1}_{\varepsilon A}+(1+\delta)\mathbf{1}_{\eta B}$ with $\delta>0$. If $k:=|B|$, since $w(A)\leq w(B)$,

$$\|x+\mathbf{1}_{\varepsilon A}\| = \|y-\mathscr{G}_k(y)\| \leq C_g^w\sigma^w_{w(B)}(y) \leq C_g^w\|y-\mathbf{1}_{\varepsilon A}\| = C_g^w\|x+(1+\delta)\mathbf{1}_{\varepsilon' B}\|.$$

Taking $\delta\searrow 0$, we obtain that the basis is w-SLC with $C_a^w\leq C_g^w$.

We prove now the right hand inequality. Assume that $\mathscr{B}$ is C_a^w-w-SLC and K_{su}-unconditional. Take $x\in\mathbb{X}$, $A=\operatorname{supp}(\mathscr{G}_m(x))$ and $z=\sum_{n\in B}a_n\mathbf{e}_n$ such that $\|x-z\|<\sigma^w_{w(A)}(x)+\varepsilon$ for $\varepsilon>0$ and $w(B)\leq w(A)$. Define $t:=\min\{|\mathbf{e}_n^*(x)| : n\in A\}$ and $\eta\equiv\{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. Now, using the item b) of Proposition 6.1.2 with $w(B\setminus A)\leq w(A\setminus B)$,

$$\begin{aligned}\|x-\mathscr{G}_m(x)\| &\leq C_a^w\|x-P_A(x)-P_{B\setminus A}(x)+t\mathbf{1}_{\eta(A\setminus B)}\|\\ &= C_a^w\|P_{(A\cup B)^c}(x-z)+t\mathbf{1}_{\eta(A\setminus B)}\|.\end{aligned}\tag{6.18}$$

Now, taking into account that $P_{(A\cup B)^c}(x-z)+t\mathbf{1}_{\eta(A\setminus B)}=T_t((x-z)-P_B(x-z))$, where T_t is the truncation operator, by (2.9),

$$\|T_t((x-y)-P_B(x-z))\|\leq K_{su}\|x-z\|. \tag{6.19}$$

By (6.18) and (6.19) the proof is over. □

6.2.2 w-almost-greedy bases

Definition 6.2.4 ([36]). An SM-basis $\mathscr{B}$ is *w-almost-greedy* if there exists a constant $C\geq 1$ such that

$$\|x-\mathscr{G}_m(x)\|\leq C\widetilde{\sigma}^w_{w(A_m(x))}(x),\ \forall x\in\mathbb{X},\forall m\in\mathbb{N}, \tag{6.20}$$

where $A_m(x)$ is the m-th greedy set of x. The least constant C in (6.20) is denoted by $C_{al}^w[\mathscr{B},\mathbb{X},w]:=C_{al}^w$ and we will say that $\mathscr{B}$ is *C_{al}^w-w-greedy*.

These bases were introduced by Dilworth et al. in [36] and they gave the following characterization.

Theorem 6.2.5 ([36]). An SM-basis $\mathscr{B}$ is w-almost-greedy if and only if $\mathscr{B}$ is w-democratic and quasi-greedy. Quantitatively,

- If $\mathscr{B}$ is C_{al}^w-w-almost-greedy, then $\mathscr{B}$ is w-democratic and quasi-greedy with

$$\max\{C_d^w,C_q\}\leq C_{al}^w.$$

- If $\mathscr{B}$ is C_d^w-w-democratic and C_q-quasi-greedy, $\mathscr{B}$ is C_{al}^w-w-almost-greedy with $C_{al}^w\leq 8C_q^4C_d^w+C_q+1$.

As for w-greedy bases, we present now the characterization of w-almost-greediness using the w-symmetry for largest coefficients since in the above theorem, is not possible to recover the constant $C_{al}^w = 1$ using $C_d^w = C_q = 1$. The following theorem can be found in [21].

Theorem 6.2.6. Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be an SM-basis of a Banach space $\mathbb{X}$. $\mathscr{B}$ is w-almost-greedy if and only if $\mathscr{B}$ is quasi-greedy and C_a^w-symmetric for largest coefficients. Quantitatively,

$$\max\{C_q, C_a^w\} \leq C_{al}^w \leq C_a^w C_q.$$

Proof. Assume that $\mathscr{B}$ is C_{al}^w-w-almost-greedy. Since

$$\|x - \mathscr{G}_m(x)\| \leq C_{al} \inf\{\|x - \sum_{n\in B} \mathbf{e}_n^*(x)\mathbf{e}_n\| : w(B) \leq w(A_m(x)), B \in \mathbb{N}^{<\infty}\},$$

we can select $B = \emptyset$. Then, we obtain that $\|x - \mathscr{G}_m(x)\| \leq C_{al}^w\|x\|$, so the basis is quasi-greedy with constant $C_q \leq C_{al}^w$.

Concerning the w-symmetry for largest coefficients, the same proof we gave in Theorem 6.2.3 actually gives $C_a^w \leq C_{al}^w$.

Assume now that $\mathscr{B}$ is C_q-quasi-greedy and C_a^w-w-SLC. Take $x \in \mathbb{X}$ and $A = \operatorname{supp}(\mathscr{G}_m(x))$. For each $\varepsilon > 0$, take B such that $\|x - P_B(x)\| < \widetilde{\sigma}_{w(A)}^w(x) + \varepsilon$ and $w(B) \leq w(A)$. Considering $t := \min\{|\mathbf{e}_n^*(x)| : n \in A\}$ and $\eta \equiv \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$, arguing in the same way than in Theorem 6.2.3,

$$\begin{aligned}
\|x - \mathscr{G}_m(x)\| &\leq C_a^w \|P_{(A\cup B)^c}(x - P_B(x)) + t\mathbf{1}_{\eta(A\setminus B)}\| \\
&= C_a^w \|T_t((\mathrm{Id}_{\mathbb{X}} - P_B)(x))\| \\
&\leq C_q C_a^w \|x - P_B(x)\|.
\end{aligned}$$

This gives the desired result. □

6.2.3 w-partially-greedy bases

Definition 6.2.7. An SM-basis $\mathscr{B}$ is *w-partially-greedy* if there exists a positive constant C such that for all m and r such that $w(\{1,...,m\}) \leq w(A_r(x))$,

$$\|x - \mathscr{G}_r(x)\| \leq C\|x - P_m(x)\|. \tag{6.21}$$

The least constant C in (6.21) is denoted by $C_p^w[\mathscr{B}, \mathbb{X}, w] := C_p^w$ and we will say that $\mathscr{B}$ is *C_p^w-w-partially-greedy*.

These type of bases were introduced the first time in [21], and we gave the following characterization based on the ideas of Theorem 2.6.2.

Theorem 6.2.8. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$. $\mathscr{B}$ is w-partially-greedy if and only if $\mathscr{B}$ is w-super-conservative and quasi-greedy. Quantitatively,

$$\max\left\{\frac{C_q}{1+\mathbf{cc}^*}, C_{sc}^w\right\} \leq C_p^w \leq 1 + 2C_q(1 + C_{sc}^w).$$

Proof. Assume that $\mathscr{B}$ is C_p^w-w-partially-greedy. To show the w-super-conservativeness, take $(A,B,\varepsilon,\varepsilon') \in \mathfrak{F}_c'(w)$ and show (6.6). Let $m = \max A$ and define the set $D = [1,..,m] \setminus A$. Of course,

$$w(\{1,...,m\}) = w(A \cup D) \leq w(B \cup D).$$

Define now $x := \mathbf{1}_{\varepsilon A} + (1+\delta)\mathbf{1}_{\varepsilon'(B\cup D)}$. Then,

$$\|\mathbf{1}_{\varepsilon A}\| = \|x - \mathscr{G}_{|B\cup D|}(x)\| \leq C_p^w \|x - P_m(x)\| = C_p^w \|(1+\delta)\mathbf{1}_{\varepsilon' B}\|.$$

Taking $\delta \searrow 0$, the basis is w-super-conservative with $C_{sc}^w \leq C_p^w$.

To prove the quasi-greediness, taking $m = 1$ in the definition of w-partially-greediness,

$$\|x - \mathscr{G}_r(x)\| \leq \|x - P_1(x)\| \leq C_p(1 + \mathbf{cc}^*)\|x\|,$$

so the basis is quasi-greedy with $C_q \leq C_p(1+\mathbf{cc}^*)$.

Now, assume that $\mathscr{B}$ is C_{sc}^w-w-super-conservative and C_q-quasi-greedy, and show that $\mathscr{B}$ is w-partially-greedy. Take $x \in \mathbb{X}$, m, and r as in the definition of w-partially-greedy, and define the following sets:

$$D = A_r(x) \cap [1,...,m],\ B = A_r(x) \setminus [1,...,m],\ A = [1,...,m] \setminus D,$$

where $A_r(x)$ is the r-th greedy set of x. Then $A_r(x) = B \cup D$ and $w(A) = w(\{1,...,m\}) - w(D) \leq w(A_r(x)) - w(D) = w(B)$. For the element x, we have the following decomposition using these sets:

$$x - \mathscr{G}_r(x) = (x - P_m(x)) - P_B(x) + P_A(x).$$

On the one hand, $\|P_B(x)\| = \|G_k(x - P_m(x))\| \leq 2C_q\|x - P_m(x)\|$ for $k := |B|$. On the other hand, using Corollary 1.3.3 and w-super-conservativeness with $w(A) \leq w(B)$,

$$\|P_A(x)\| \leq \max_{i\in A} |\mathbf{e}_i^*(x)| \sup_{\varepsilon \in \Psi_A} \|\mathbf{1}_{\varepsilon A}\| \leq C_{sc}^w \max_{i\in A} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\|,$$

for any $\eta \in \Psi_B$. Now, by 2.1.12 with $\eta \equiv \{\mathrm{sign}(\mathbf{e}_n^*(x))\}$,

$$\begin{aligned}
\|P_A(x)\| &\leq C_{sc}^w \max_{i\in A} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\| \leq C_{sc}^w \min_{i\in B} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta B}\| \\
&= C_{sc}^w \min_{i\in B} |\mathbf{e}_i^*(x - P_m(x))| \, \|\mathbf{1}_{\eta B}\| \\
&\leq 2C_q C_{sc}^w \|x - P_m(x)\|.
\end{aligned}$$

Then, $\|x - \mathscr{G}_r(x)\| \leq (1 + 2C_q + 2C_q C_{sc}^w)\|x - P_m(x)\|$. □

6.2.4 *w*-semi-greedy bases

The last w-greedy-type bases that we study are the w-semi-greedy bases, that were introduced in [36]. We remind the definition of the Thresholding Chebyshev Greedy Algorithm that can be found in Chapter 5. If $A_m(x)$ is the m-th greedy set of an element $x \in \mathbb{X}$, $\mathfrak{CG}_m(x)$ is any element in $\langle \mathbf{e}_i : i \in A_m(x)\rangle$ such that

$$\|x - \mathfrak{C}\mathfrak{G}_m(x)\| = \min\left\{\left\|x - \sum_{i\in A_m(x)} a_i\mathbf{e}_i\right\| : a_i \in \mathbb{F}\right\}.$$

Definition 6.2.9 ([36])**.** An SM-basis $\mathscr{B}$ in a Banach space $\mathbb{X}$ is *w-semi-greedy* if there exists a positive constant C such that

$$\|x - \mathfrak{C}\mathfrak{G}_m(x)\| \leq C\sigma^w_{w(A_m(x))}(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{6.22}$$

The least constant C in (6.22) is denoted by $C^w_{sg}[\mathscr{B},\mathbb{X},w] := C^w_{sg}$ and we will say that $\mathscr{B}$ is C^w_{sg}*-w-semi-greedy*.

In [36], the authors proved the following result that is the extension of Theorem 5.0.3 for the weighted case.

Theorem 6.2.10 ([36])**.** Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

- If $\mathscr{B}$ is C^w_d-w-democratic and C_q-quasi-greedy, then $\mathscr{B}$ is w-semi-greedy with

$$C^w_{sg} \leq 1 + 3C_q + 16C_q^3 C^w_d.$$

- If $\mathbb{X}$ has finite cotype and $\mathscr{B}$ is a Schauder w-semi-greedy basis, then $\mathscr{B}$ is w-almost-greedy.

In the following theorem that can be found in [22], we show that the condition of the finite cotype over $\mathbb{X}$ can be removed.

Theorem 6.2.11. Assume that $\mathscr{B}$ is an SM-basis in a Banach space $\mathbb{X}$.

a) If $\mathscr{B}$ is C_q-quasi-greedy and C^w_s-w-super-democratic, then $\mathscr{B}$ is C^w_{sg}-w-semi-greedy with constant $C^w_{sg} \leq C_q + 4C_qC^w_s$.

b) If $\mathscr{B}$ is C^w_{sg}-w-semi-greedy and Schauder, then $\mathscr{B}$ is C^w_s-w-super-democratic and C_q-quasi-greedy.

Remark 6.2.12. The bound $C^w_{sg} = O(C_qC^w_s)$ it is an improvement with respect to the bound of $C^w_{sg} = O(C_q^3C^w_d)$ proved by Dilworth et al. in Theorem 6.2.10. To show this fact, we only have to use Proposition 6.1.9 and conclude that $C^w_s \leq 4\kappa^2 C^w_d C_q$, where κ as in 6.1.9.

Before proving Theorem 6.2.11, we give a technical lemma that we will use in the proof.

Lemma 6.2.13. Assume that $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ is an SM-Schauder basis of a Banach $\mathbb{X}$ with $\mathfrak{K}_b$ the basis constant and C^w_{sg}-w-semi-greedy. Then, if $x \in \mathbb{X}$ with finite support, F is a set such that $F > \operatorname{supp}(x)$ and $w(F) \leq w(G)$ for some greedy set G, then

$$\min_{i\in G} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta F}\| \leq C^w_{sg}(1+\mathfrak{K}_b)\|x\|,\ \forall \eta \in \Psi_F.$$

Proof. Take $x \in \mathbb{X}$ with finite support, G a greedy set of x with $k := |G|$, F and $\eta \in \Psi_B$ as in the statement of this lemma. Define the following element

$$y := \min_{i \in G} |\mathbf{e}_i^*(x)| \mathbf{1}_{\eta F} + (\mathrm{Id}_{\mathbb{X}} - P_G)(x) + \sum_{i \in G} (\mathbf{e}_i^*(x) + \delta \varepsilon_i) \mathbf{e}_i,$$

where $\delta > 0$ and $\varepsilon \equiv \{\mathrm{sign}(\mathbf{e}_j^*(x))\}$. Then, for the element y, the set G is the k-th greedy set for y. Hence, applying the TCGA, there exists a sequence $(a_i)_{i \in G}$ such that

$$y - \mathfrak{C}\mathfrak{G}_k(y) := \min_{i \in G} |\mathbf{e}_i^*(x)| \mathbf{1}_{\eta F} + (\mathrm{Id}_{\mathbb{X}} - P_G)(x) + \sum_{i \in G} a_i \mathbf{e}_i.$$

Hence,

$$\begin{aligned}
\min_{i \in G} |\mathbf{e}_i^*(x)| \|\mathbf{1}_{\eta F}\| &\leq (1 + \mathfrak{K}_b) \| \min_{i \in G} |\mathbf{e}_i^*(x)| \mathbf{1}_{\eta F} + (\mathrm{Id}_{\mathbb{X}} - P_G)(x) + \sum_{i \in G} a_i \mathbf{e}_i \| \\
&\leq (1 + \mathfrak{K}_b) C_{sg}^w \sigma_{w(G)}^w (y) \\
&\leq (1 + \mathfrak{K}_b) C_{sg}^w \| y - \min_{i \in G} |\mathbf{e}_i^*(x)| \mathbf{1}_{\eta F} \| \\
&= (1 + \mathfrak{K}_b) C_{sg}^w \| (\mathrm{Id}_{\mathbb{X}} - P_G)(x) + \sum_{i \in G} (\mathbf{e}_i^*(x) + \delta \varepsilon_i) \mathbf{e}_i \|.
\end{aligned}$$

Taking limits when δ goes to 0, we obtain the result. □

Proof of Theorem 6.2.11. First, we show the proof of a). Suppose that $\mathscr{B}$ is C_q-quasi-greedy and C_{sd}^w-w-super-democratic. To show the w-semi-greediness, we will follow the same procedure as in the proof of [35, Theorem 4.1] and [31, Theorem 3.2]. Take $x \in \mathbb{X}$ with $A = \mathrm{supp}(\mathscr{G}_m(x))$ and $z = \sum_{i \in B} a_i \mathbf{e}_i$ with $w(B) \leq w(A_m(x))$ such that $\|x - z\| < \sigma_{w(A)}(x) + \delta$, for $\delta > 0$. We write $x - z := \sum_{i=1}^{\infty} y_i \mathbf{e}_i$, where $y_i = \mathbf{e}_i^*(x) - a_i$ for $i \in B$ and $y_i = \mathbf{e}_i^*(x)$ for $i \notin B$. To prove that $\mathscr{B}$ is w-semi-greedy we only have to show that there exists $\omega \in \mathbb{X}$ so that $\mathrm{supp}(x - \omega) \subset A$ and $\|\omega\| \leq c\|x - z\|$ for some positive constant c. If $\alpha = \max_{j \notin A} |\mathbf{e}_j^*(x)|$, we take the element ω as is defined in [31]:

$$\omega := \sum_{i \in A} T_\alpha(y_i) \mathbf{e}_i + (\mathrm{Id}_{\mathbb{X}} - P_A)(x) = \sum_{i=1}^{\infty} T_\alpha(y_i) \mathbf{e}_i + \sum_{i \in B \setminus A} (\mathbf{e}_i^*(x) - T_\alpha(y_i)) \mathbf{e}_i.$$

Of course, ω satisfies that $\mathrm{supp}(x - \omega) \subset A$ and we show that $\|\omega\| \leq (C_q + 4C_q C_s^w) \|x - z\|$. To obtain this bound, using Lemma 2.1.13,

$$\left\| \sum_{i=1}^{\infty} T_\alpha(y_i) \mathbf{e}_i \right\| \leq C_q \|x - z\|. \tag{6.23}$$

Now, if we take $\Lambda := \{n : |\mathbf{e}_n^*(x - z)| \geq \min_{i \in A \setminus B} |\mathbf{e}_i^*(x)|, n \notin A \setminus B\}$, $C = \Lambda \cup (A \setminus B)$ is a greedy set of $x - z$ and $w(B \setminus A) \leq w(A \setminus B) \leq w(C)$. Thus, taking into account that $|\mathbf{e}_i^*(x) - T_\alpha(y_i)| \leq 2\alpha$ for $i \in B \setminus A$, using Corollary 4.2.5,

$$\left\| \sum_{i \in B \setminus A} (\mathbf{e}_i^*(x) - T_\alpha(y_i)) \mathbf{e}_i \right\| \leq 2\alpha \sup_{\varepsilon' \in \Psi_{B \setminus A}} \|\mathbf{1}_{\varepsilon'(B \setminus A)}\|. \tag{6.24}$$

Now, using w-super-democracy and Lemma 2.1.14 with $\eta \equiv \{\operatorname{sign}(\mathbf{e}_j^*(x-z))\}$,

$$\begin{aligned}
2\alpha \sup_{\varepsilon' \in \Psi_{B\setminus A}} \|\mathbf{1}_{\varepsilon'(B\setminus A)}\| &\leq 2\alpha C_s^w \|\mathbf{1}_{\eta C}\| \leq 2C_s^w \min_{i\in C} |\mathbf{e}_i^*(x-z)| \|\mathbf{1}_{\eta C}\| \\
&\leq 4C_q C_s^w \|x-z\|. \qquad (6.25)
\end{aligned}$$

Then, by (6.23), (6.24) and (6.25) we obtain the result.

Prove now b). Assume that $\mathscr{B}$ is C_{sg}^w-w-semi-greedy. To prove this part, in [36], the authors proved Proposition 6.1.7 under the assumption of w-semi-greediness. Due to Proposition 6.1.7, we only consider $\sum_{n=1}^{\infty} w_n = \infty$ and $\sup_n w_n < \infty$ and, also, we consider the following bound given in [36]: for any finite set A with $w(A) \leq \limsup_{n\to+\infty} w_n$,

$$\varphi_u(|A|) \leq 2C_{sg}^w \mathfrak{K}_b. \qquad (6.26)$$

Quasi-greediness: to show that $\mathscr{B}$ is quasi-greedy, take $x \in \mathbb{X}$ such that $|\operatorname{supp}(x)| < \infty$, and due to the homogeneity, we can assume that $\max_j |\mathbf{e}_j^*(x)| \leq 1$, $m \in \mathbb{N}$ and consider that $w(A) > \limsup_{n\to\infty} w_n$ where $A = \operatorname{supp}(\mathscr{G}_m(x))$. With the same argument than in Proposition 6.1.9, we can take E and $n_0 \in \mathbb{N}$ with $E > \operatorname{supp}(x)$ and $n_0 > \max E$ such that

$$w(E) \leq w(A) < w(E) + w_{n_0}.$$

Set $F := E \cup \{n_0\}$ and $\alpha = \min_{j\in A} |\mathbf{e}_j^*(x)|$. Define the element

$$y := (x - \mathscr{G}_m(x)) + (\alpha+\lambda)\mathbf{1}_F,$$

with $\lambda > 0$. Hence, the greedy set of y is F and then, if the scalars $(a_n)_n$ are given by the TCGA and $\varepsilon \equiv \{\operatorname{sign}(\mathbf{e}_j^*(x))\}$,

$$\begin{aligned}
\|x - \mathscr{G}_m(x)\| &\leq \mathfrak{K}_b \|x - \mathscr{G}_m(x) + \sum_{n\in F} a_n \mathbf{e}_n\| \leq \mathfrak{K}_b C_{sg}^w \sigma_{w(F)}^w(y) \\
&\leq C_{sg}^w \mathfrak{K}_b \|(x - \mathscr{G}_m(x)) + \sum_{i\in A} (\mathbf{e}_i^*(x) + \lambda\varepsilon_i)\mathbf{e}_i + (\alpha+\lambda)\mathbf{1}_F\| \\
&= C_{sg}^w \mathfrak{K}_b \|x + (\alpha+\lambda)\mathbf{1}_F + \lambda \mathbf{1}_{\varepsilon A}\|.
\end{aligned}$$

Taking limits when λ goes to 0,

$$\|x - \mathscr{G}_m(x)\| \leq C_{sg}^w \mathfrak{K}_b (\|x\| + \|\alpha \mathbf{1}_E\| + \alpha\|\mathbf{e}_{n_0}\|). \qquad (6.27)$$

Since $\alpha\|\mathbf{e}_{n_0}\| \leq \mathbf{c}\mathbf{c}^*\|x\|$, we only have to estimate $\|\alpha\mathbf{1}_E\|$. For that, using Lemma 6.2.13,

$$\|\alpha\mathbf{1}_E\| \leq (1+\mathfrak{K}_b)C_{sg}^w\|x\|.$$

Then, we have that the basis is quasi-greedy for elements with finite support with

$$\|x - \mathscr{G}_m(x)\| \leq C_{sg}^w \mathfrak{K}_b (1 + (1+\mathfrak{K}_b)C_{sg}^w + \mathbf{c}\mathbf{c}^*)\|x\|.$$

Define now $C_1 = C_{sg}^w \mathfrak{K}_b (1 + (1+\mathfrak{K}_b)C_{sg}^w + \mathbf{c}\mathbf{c}^*)$. To show the quasi-greediness for any $x \in \mathbb{X}$, we will use the Lemma 2.0.3: let A be the m-th greedy set of $x \in \mathbb{X}$. Then, for any $\varepsilon > 0$, there exists an element $y \in \mathbb{X}$ with finite support such that $\|x-y\| < \varepsilon$ and A is the m-th greedy set of

y. Hence, if $\mathcal{G}_m(x) = P_A(x)$ and $\mathcal{G}_m(y) = P_A(y)$,

$$\begin{aligned} \|x - \mathcal{G}_m(x)\| &\leq \|x-y\| + \|y - \mathcal{G}_m(y)\| + \|\mathcal{G}_m(y) - \mathcal{G}_m(x)\| \\ &\leq \varepsilon + C_1\|y\| + \|P_A(x-y)\| \\ &\leq \varepsilon + C_1\varepsilon + C_q\|x\| + \|P_A\|\varepsilon \\ &= \varepsilon(1 + C_1 + \|P_A\|) + C_1\|x\|. \end{aligned}$$

If ε goes to 0, we obtain that $\mathscr{B}$ is C_q-quasi-greedy with $C_q \leq C_1$.

Now, consider that $w(A) \leq \limsup_{n\to\infty} w_n$. Using (6.26),

$$\max_{\varepsilon\in\Psi_A} \|\mathbf{1}_{\varepsilon A}\| \leq 2C_{sg}^w \mathfrak{K}_b.$$

Then, using Corollary 1.3.3,

$$\|\mathcal{G}_m(x)\| \leq \max_j |\mathbf{e}_j^*(x)| 2C_{sg}^w \mathfrak{K}_b \leq 2\mathbf{c}^* C_{sg}^w \mathfrak{K}_b \|x\|.$$

Hence, $\mathscr{B}$ is C_q-quasi-greedy with $C_q \leq 1 + 2\mathbf{c}^* C_{sg}^w \mathfrak{K}_b$.

w-super-democracy: to prove that $\mathscr{B}$ is w-super-democratic, take $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}'(w)$ and show (6.3). If $w(B) > \limsup_{n\to\infty} w_n$, we tan take the set F as before, that is, $F = E \cup \{n_0\}$ such that $w(E) \leq w(B) < w(F)$, $n_0 > \max E$ and $E > A \cup B$. Then, taking the element $x = \mathbf{1}_{\varepsilon A} + (1 + \delta)\mathbf{1}_F$, with $\delta > 0$, the k-th greedy set of x is F with $k := |F|$. Using the scalars $(a_i)_{i\in F}$ given by the TCGA, we have that

$$\|\mathbf{1}_{\varepsilon A}\| \leq \mathfrak{K}_b \left\| \mathbf{1}_{\varepsilon A} + \sum_{i\in F} a_i \mathbf{e}_i \right\| \leq \mathfrak{K}_b C_{sg}^w \sigma_{w(F)}^w(x) \leq \mathfrak{K}_b C_{sg}^w \|(1+\delta)\mathbf{1}_F\|.$$

Taking $\delta \searrow 0$, $\|\mathbf{1}_{\varepsilon A}\| \leq \mathfrak{K}_b C_{sg}^w (\|\mathbf{1}_E\| + \mathbf{c}\mathbf{c}^* \|\mathbf{1}_{\eta B}\|)$ since $\|\mathbf{e}_{n_0}\| \leq \mathbf{c}\mathbf{c}^* \|\mathbf{1}_{\eta B}\|$. Now, as $E > B$ and $w(B) \geq w(E)$, using Lemma 6.2.13, we obtain that

$$\|\mathbf{1}_E\| \leq (1 + \mathfrak{K}_b) C_{sg}^w \|\mathbf{1}_{\varepsilon' B}\|.$$

Hence, the basis is w-super-democratic with constant $C_s^w \leq \mathfrak{K}_b C_{sg}^w ((1 + \mathfrak{K}_b) C_{sg}^w + \mathbf{c}\mathbf{c}^*)$.

Now, if $w(B) \leq \limsup_{n\to\infty} w_n$, since $\max_{\varepsilon\in\Psi_C} \|\mathbf{1}_{\varepsilon C}\| \leq 2\mathfrak{K}_b C_{sg}^w$ for any finite set by (6.26),

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2\mathbf{c}^* \mathfrak{K}_b C_{sg}^w \|\mathbf{1}_{\varepsilon' B}\|.$$

□

6.3 An example of a w-greedy basis

This section is dedicated to a particular example of which we study its w-greedy type properties for different weights.

Let $w = (w_n)_{n=1}^{\infty} \in (0,\infty)^{\mathbb{N}}$ with $w_n \leq 1$ for every $n \in \mathbb{N}$. For a fixed $1 \leq p < q \leq \infty$, we define the Banach space $\mathbb{X} = \ell_q \cap \ell_p(w)$, where if $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is a sequence,

$$\|\mathbf{a}\| = \max\{\|\mathbf{a}\|_{\ell_q}, \|\mathbf{a}\|_{\ell_p(w)}\},$$

where $\|\mathbf{a}\|_{\ell_p(w)} = \left(\sum |a_n|^p w_n\right)^{1/p}$. If $q = \infty$, we take c_0 instead of ℓ_∞. We will denote by $\mathscr{B}$ the canonical basis in the space $\mathbb{X}$. Also, this space is called the **Rosenthal-Woo space** (see [88]).

Proposition 6.3.1. If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is the canonical basis in $c_0 \cap \ell_p(w)$, $\mathscr{B}$ is w-greedy with $C_g^w = 1$.

Proof. Clearly it is unconditional with constant 1. To prove that the canonical basis is w-greedy with constant 1, we need to show that $\mathscr{B}$ has the w-symmetric for largest coefficients with constant 1 (Theorem 6.2.3). For that, take $(A, B, \varepsilon, \varepsilon', x) \in \mathfrak{F}_d(w)$ and show (6.7):

$$\begin{aligned} \|x + \mathbf{1}_{\varepsilon A}\| &= 1 \vee \left(\sum_{n \in \operatorname{supp}(x)} |\mathbf{e}_n^*(x)|^p w_n + w(A) \right)^{1/p} \\ &\leq 1 \vee \left(\sum_{n \in \operatorname{supp}(x)} |\mathbf{e}_n^*(x)|^p w_n + w(B) \right)^{1/p} = \|x + \mathbf{1}_{\varepsilon' B}\|. \end{aligned}$$

Then, the basis satisfies the w-symmetric for largest coefficients with $C_a^w = 1$ and hence, using b) of Theorem 6.2.3, the basis is w-greedy with $C_g^w = 1$. □

Proposition 6.3.2. Let $1 \leq p < q < \infty$. If $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is the canonical basis in $\mathbb{X}$ and w is decreasing with $w_k \to 0$ when $k \to +\infty$, then:

a) $\mathscr{B}$ is w-super-conservative.

b) $\mathscr{B}$ is not w-democratic.

c) $\mathscr{B}$ is not democratic if $\dfrac{\left(\sum_{k=1}^{n} w_k\right)^{1/p}}{n^{1/q}} \to \infty$.

Proof. First of all, we show a). Select $(A, B, \varepsilon, \varepsilon') \in \mathfrak{F}_c'(w)$. Since w is decreasing, we must have $|A| \leq |B|$. Thus,

$$\|\mathbf{1}_{\varepsilon A}\| = (|A|)^{1/q} \vee w(A)^{1/p} \leq (|B|)^{1/q} \vee w(B)^{1/p} = \|\mathbf{1}_{\varepsilon' B}\|.$$

Hence, the basis is w-super-conservative.

b) To show that the basis is not w-democratic, take M and k natural numbers and define $A = \{1, ..., M\} + k$. It is clear that

$$C_d^w \geq \frac{\|\mathbf{1}_A\|}{\|\mathbf{e}_1\|} \geq M^{1/q}, \tag{6.28}$$

if $w(A) \leq w_1$. Since w is decreasing,

$$w(A) = \sum_{n=1}^{M} w_{n+k} \leq M w_{k+1}.$$

Fix now k and select M_k the largest integer such that $M_k w_{k+1} \leq w_1$. Thus, $2M_k > \frac{w_1}{w_{k+1}}$. Using this k and M_k in (6.28),

$$C_d^w \geq \left(\frac{w_1/2}{w_{k+1}}\right)^{1/q} \to \infty,$$

since $w_k \to 0$.

c) Finally, to show that the basis is not democratic, given n and k observe that

$$\frac{\|\mathbf{1}_{\{1,\dots,n\}}\|}{\|\mathbf{1}_{\{1,\dots,n\}+k}\|} \geq \frac{(\sum_{k=1}^n w_k)^{1/p}}{\max\{n^{1/q}, (\sum_{j=1}^n w_{j+k})^{1/p}\}}.$$

Since w is decreasing and $w_k \to 0$, take k such that $\max\{n^{1/q}, (\sum_{j=1}^n w_{j+k})^{1/p}\} = n^{1/q}$. Taking this k, by our assumption, the basis is not democratic. □

Remark 6.3.3. Some examples of weights that verify the item c) of the above proposition are the following:

- $w_n = n^{-\theta}$, with $\theta < 1 - p/q$.
- $w_n = \frac{\ln(n)^\gamma}{n^{1-p/q}}$ with $\gamma > 0$.

6.4 Open questions

In Theorem 5.2.1 of Chapter 5 we have studied the characterization of semi-greediness under the condition of ρ-admissible bases, that is a weaker condition respect to the condition of Schauder bases. In this chapter, we prove the corresponding weighted version of this theorem (Theorem 6.2.11) but assuming that $\mathscr{B}$ is Schauder. The reason is because the definition of ρ-admissibility concerns about the cardinality of sets.

Question 1: is it possible to give a similar definition of ρ-admissibility for weights showing that this new condition is weaker than Schauder and prove Theorem 6.2.11?

Question 2: is it possible to prove Theorem 6.2.11 for general SM-bases?

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